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OF

THE LONDON MATHEMATICAL SOCIETY.

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PROCEEDINGS

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THE LONDON MATHEMATICAL SOCIETY.

VOL. XXXII.

THIRTY-SIXTH SESSION, 1899-1900

(since the Formation of the Society, January 16th, 1865).

Thursday, January 11th, 1900.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President, in the Chair.

Ten members present.

The following were elected members of the Society:—Miss Beatrice Mabel Cave Browne Cave, Miss Frances Evelyn Cave Browne Cave, and Mr. Herbert William Richmond, M.A., Fellow of King's College, Cambridge.

Prof. Love gave a sketch of a paper by Mr. J. H. Michell, entitled "Elementary Distributions of Plane Stress."

Lt.-Col. Cunningham (Mr. Kempe in the Chair) communicated a preliminary sketch of "A General Method of Factorization of Biquadratics" with special application to Quartans, $N = x^4 + y^4$.

The following papers were communicated in abstract, viz.:—

A Problem in Resonance, illustrative of the Mechanical Theory of Selective Absorption of Light, by Prof. Lamb.

An Abstract Simple Group of Order 25920, by Dr. L. E. Dickson.

The following presents were made to the Library:—

"Educational Times," January, 1900.

"Indian Engineering," Vol. xxvi., Nos. 22–25, Nov. 25–Dec. 16, 1899.

"Annales de la Faculté des Sciences," Série 2, Tome i., Fasc. 1, 2; Toulouse, 1899.

Ostwald, W.—"Periodische Erscheinungen bei der Auflösung des Chroms in Säuren," 8vo; Leipzig, 1899.

Garten, Dr. S.—"Beiträge zur Physiologie des elektrischen Organes des Zitterrochen," 8vo; Leipzig, 1899.

Acta Societatis Scientiarum Fennicæ:—

Lindelöf, E.—"Remarques sur un Principe général de la Théorie des Fonctions Analytiques," Tom xxiv., No. 7, April, 1898.

Mellin, H. J.—"Ueber eine Verallgemeinerung der Riemannschen Function $\zeta(s)$," Tom xxiv., No. 10.

The following exchanges were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxiii., St. 11; Leipzig, 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. vi., No. 3; New York, December, 1899.

"Journal für die reine und angewandte Mathematik," Bd. cxxi., Heft 3; Berlin, December, 1899.

"Annali di Matematica," Serie 3, Tomo iii., Fasc. 3, 4; Milano, Dicembre, 1899.

"Archives Néerlandaises," Série 2, Tome iii., Livr. 2; La Haye, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. viii., Fasc. 11, 12; Roma, 1899.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," No. 5; 1899.

"Proceedings of the Physical Society," Vol. xvi., Pts. 7, 8; December, 1899.

"Periodico di Matematica," Serie ii., Vol. ii., Fasc. 3; Livorno, Nov.-Dec., 1899. Supplemento, 1899, Fasc. 2.

An Abstract Simple Group of Order 25920. *By* L. E. Dickson, Ph.D.

Received and communicated January 11th, 1900.

1. The abstract form of the known simple group of order 25920 was determined by the writer in the *Proceedings* of the Society, Vol. xxxi., pp. 40–45. The present paper gives a simpler method of solving the problem. Furthermore, it has direct contact with the developments of Jordan on the group of the equation for the 27 lines on a cubic surface.* We first set up the abstract form of a sub-group of order 960.

2. Theorem.—*The abstract group G_{60} generated by the operators E_1, E_2, E_3 with the generational relations*

$$(1)\quad E_1^3 = E_2^2 = E_3^2 = I, \quad (E_1E_2)^3 = (E_2E_3)^3 = (E_1E_3)^2 = I$$

is put into holohedric isomorphism with a linear group L_{60} by the following correspondence of generators :—

$$(2)\quad \begin{cases} E_1 \sim \xi_1' = \xi_2, & \xi_2' = \xi_3, \quad \xi_3' = \xi_1, \quad \xi_4' = \xi_4, \quad \xi_5' = \xi_5, \\ E_2 \sim \xi_1' = \xi_2, & \xi_2' = \xi_1, \quad \xi_3' = \xi_4, \quad \xi_4' = \xi_3, \quad \xi_5' = \xi_5, \\ E_3 \sim \xi_1' = \xi_2, & \xi_2' = \xi_1, \quad \xi_3' = \xi_3, \quad \xi_4' = \xi_5, \quad \xi_5' = \xi_4. \end{cases}$$

Indeed, the abstract group is holohedrically isomorphic with the alternating group on five letters.†

3. Theorem.—*The abstract group G_{16} generated by the operators B_1, B_2, B_3, B_4 with the generational relations*

$$(3)\quad B_i^2 = I, \quad B_iB_j = B_jB_i \quad (i, j = 1, 2, 3, 4)$$

is holohedrically isomorphic with the commutative linear group

$$L_{16} = \{I,\ C_1C_2,\ C_1C_3,\ C_1C_4,\ C_1C_5,\ C_2C_3,\ C_2C_4,\ C_2C_5,\ C_3C_4,\ C_3C_5,\ C_4C_5,$$
$$C_1C_2C_3C_4,\ C_1C_2C_3C_5,\ C_1C_2C_4C_5,\ C_1C_3C_4C_5,\ C_2C_3C_4C_5\},$$

---

[*] Jordan, *Traité des Substitutions*, pp. 316–329; Dickson, *Comptes Rendus*, Vol. cxxviii. (April, 1899), pp. 873–875.

[†] *Moore, Proc. Lond. Math. Soc.*, Vol. xxviii., pp. 357–366.

where C_i *denotes the linear substitution*

$$C_i: \quad \xi_i' = -\xi_i, \quad \xi_j' = \xi_j \quad (j = 1, \ldots, 5;\ j \neq i).$$

We may set up the following correspondence of generators:—

$$(4) \quad B_1 \sim C_1C_2, \quad B_2 \sim C_2C_3, \quad B_3 \sim C_3C_4, \quad B_4 \sim C_4C_5.$$

4. THEOREM.—*The abstract group generated by the operators* E_1, E_2, E_3, B_1, B_2, B_3, B_4 *with the generational relations* (1), (3), *and*

$$(5) \quad \left\{ \begin{array}{lll} E_1^{-1}B_1E_1 = B_1B_2, & E_1^{-1}B_2E_1 = B_1, & E_1^{-1}B_3E_1 = B_2B_3, \\ & & E_1^{-1}B_4E_1 = B_4, \\ E_2^{-1}B_1E_2 = B_1, & E_2^{-1}B_2E_2 = B_1B_2B_3, & E_2^{-1}B_3E_2 = B_3, \\ & & E_2^{-1}B_4E_2 = B_3B_4, \\ E_3^{-1}B_1E_3 = B_1, & E_3^{-1}B_2E_3 = B_1B_2, & E_3^{-1}B_3E_3 = B_3B_4, \\ & & E_3^{-1}B_4E_3 = B_4, \end{array} \right.$$

is of order 960, *and is holohedrically isomorphic with the linear group* L_{960} *on the indices* $\xi_1, \ldots, \xi_5$, *given as the product of the permutable groups* L_{16} *and* L_{60}.

The permutable groups L_{16} and L_{60}, having only the identical substitution in common, generate a group of order 16.60. The isomorphic groups G_{16} and G_{60}, subject to the relations (5), generate an isomorphic group G_{960}.

5. THEOREM.—*The abstract group* G_{960} *is generated by the operators* E_1, E_2, E_3, B_1 *subject to the generational relations*

$$(6) \quad E_1^3 = E_2^2 = E_3^2 = B_1^2 = I, \quad (E_1E_2)^3 = (E_2E_3)^3 = (B_1E_1)^3 = I,$$
$$(E_1E_3)^2 = (B_1E_2)^2 = (B_1E_3)^2 = I.$$

From the relations (5) we derive at once the following:—

$$(7) \quad B_2 = E_1B_1E_1^2, \quad B_3 = E_1E_2E_1^2B_1E_1E_2E_1^2, \quad B_4 = E_2E_3B_3E_3E_2.$$

The relation $\qquad E_1^{-1}B_1E_1 = B_1B_2$

then requires $\qquad (B_1E_1)^3 = I.$

Also, by (5), B_1 is commutative with E_2 and E_3. It follows that the operators E_1, E_2, E_3, B_1 of G_{960} are subject to the relations (6). In order to prove that they are subject to no relations independent of

(6), we proceed to show that the relations (5) can be derived from (6) and (7). Noting then that B_2, B_3, B_4 are expressed in terms of E_1, E_2, E_3, B_1 by (7), our theorem will be proven.

$$B_1B_2 = B_1E_1B_1E_1^2 = E_1^2B_1E_1 \quad [\text{since } B_1E_1 \text{ is of period } 3].$$

$$B_1B_2B_3 = E_1^2B_1E_1 . B_3 = E_1^2B_1E_1^2E_2E_1^2B_1E_1E_2E_1^2.$$

Replacing $E_1^2E_2E_1^2$ by $E_2E_1E_2$ and interchanging B_1E_2 with E_2B_1,

$$\begin{aligned} B_1B_2B_3 &= E_1^2E_2B_1E_1B_1E_2B_1E_2E_1^2 = E_1^2E_2B_1E_1B_1E_1^2E_2E_1 \\ &= E_1^2E_2E_1^2B_1E_1E_2E_1 = E_2E_1E_2B_1E_2E_1^2E_2 \\ &\qquad\qquad [\text{since } B_1E_1 \text{ is of period } 3] \\ &= E_2E_1B_1E_1^2E_2 = E_2^{-1}B_2E_2. \end{aligned}$$

$$E_3^{-1}B_2E_3 = E_3E_1B_1E_1^2E_3 = E_1^2E_3B_1E_3E_1 = E_1^2B_1E_1 = B_1B_2.$$

$$B_2B_3 = E_1B_1E_2E_1^2B_1E_1E_2E_1^2 = E_1E_2E_1B_1E_1^2E_2E_1^2,$$

upon setting $\quad B_1E_2 = E_2B_1, \quad B_1E_1^2B_1E_1 = E_1B_1E_1^2.$

Hence $\quad B_2B_3 = E_2E_1^2E_2B_1E_2E_1E_2 = E_2E_1^2B_1E_1E_2 = E_1^{-1}B_3E_1.$

$$E_2^{-1}B_3E_2 = E_2E_1E_2E_1^2B_1E_1E_2E_1^2E_2 = E_1E_2E_1^2\,(E_2B_1E_2)\,E_1E_2E_1^2 = B_3,$$

since $$E_2E_1E_2E_1^2 = E_1E_2E_1^2E_2.$$

$$B_3B_4 = E_1E_2E_1^2B_1E_1E_2E_1^2 . E_2E_3 . E_1E_2E_1^2B_1E_1E_2E_1^2 . E_3E_2.$$

But $$\begin{aligned} E_1E_2E_1^2E_2E_3E_1E_2E_1^2 &= E_1^2E_2E_1E_3E_1E_2E_1^2 = E_1^2E_2E_3E_2E_1^2 \\ &= E_1^2E_3E_2E_3E_1^2 = E_3E_1E_2E_1E_3 \\ &= E_3E_2E_1^2E_2E_3. \end{aligned}$$

Hence $$\begin{aligned} B_3B_4 &= E_1E_2E_1^2B_1\,(E_3E_2E_1^2E_2E_3)\,B_1E_1E_2E_1^2E_3E_2 \\ &= E_1E_2E_1^2E_3E_2B_1E_1^2B_1E_2E_3E_1E_2E_3E_1E_2 \\ &= E_1E_2E_1^2E_3E_2E_1B_1E_1E_2E_1^2E_3E_2E_3E_1E_2 \\ &= E_1E_2E_3E_1E_2E_1B_1E_1E_2E_1^2E_2E_3E_2E_1E_2 \\ &= E_1E_2E_3E_2E_1^2\,(E_2B_1E_2)\,E_1E_2E_1^2E_3E_1^2E_2E_1^2 \\ &= E_1E_3E_2E_1\,(E_3E_2B_1E_2E_3)\,E_1^2E_2E_3E_1^2 \\ &= E_3E_1E_2E_1^2\,(E_2E_3E_2B_1E_2E_3E_2)\,E_1E_2E_1^2E_3 \\ &= E_3^{-1}\,(E_1E_2E_1^2B_1E_1E_2E_1^2)\,E_3 = E_3^{-1}B_3E_3. \end{aligned}$$

Using the last result, we find

$$E_2^{-1}B_4E_2 \equiv E_3B_3E_3 = B_3B_4.$$

To prove that
$$E_1^{-1}B_4E_1 = B_4,$$
we note that
$$E_1^{-1}(E_3B_3E_3)E_1 = E_1^{-1}(B_3B_4)E_1,$$
or
$$E_3E_1B_3E_1^2E_3 = B_2B_3 . E_1^{-1}B_4E_1.$$

But the first product equals

$$E_3B_1B_2B_3E_3 = B_2B_3B_4.$$

Finally, $E_3^{-1}B_4E_3 = E_3E_2E_3B_3E_3E_2E_3 = E_2E_3(E_2B_3E_2)E_3E_2 = B_4.$

6. The general orthogonal group on five indices with coefficients taken modulo 3 has a sub-group O_{25920} given by the extension of the group L_{960} by the following substitution w of period 3:—*

$$\xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4, \quad \xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4,$$
$$\xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4, \quad \xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4.$$

We note that w^2 and w^{-1} both have the form

$$\xi_1' = \quad \xi_1 + \xi_2 + \xi_3 + \xi_4, \quad \xi_2' = -\xi_1 - \xi_2 + \xi_3 + \xi_4,$$
$$\xi_3' = -\xi_1 + \xi_2 - \xi_3 + \xi_4, \quad \xi_4' = -\xi_1 + \xi_2 + \xi_3 - \xi_4.$$

If (12)(34) denote the linear substitution (2) corresponding to E_2, we readily verify that

$$C_3C_4 = w(12)(34)w^{-1}.$$

Hence the simple group O_{25920} is generated by the linear substitutions (2) (which generate L_{60}) together with w.

7. THEOREM.† — *The abstract group O generated by the operators* E_1, E_2, E_3, B_1, W *subject to the generational relations* (6) *and*

(8) $W^3 = 1,\ W^{-1}E_2W = B_1E_2,\ W^{-1}E_1W = B_1E_1,\ W^{-1}B_1W = B_3B_1E_2,$

(9) $$WB_4 = B_4B_2E_1E_2E_1^2W^2,$$

(10) $$(WE_3E_2E_1W)E_3 = E_1^2E_2E_3E_2E_1(WE_3E_2E_1W),$$

is simply isomorphic with the linear group O_{25920}.

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* *American Journal of Mathematics*, Vol. XXI., pp. 193–256.

† *For simplicity* B_2, B_3, and B_4 have been retained in the formulæ, but are, in *fact, to be eliminated by (7).*

In virtue of the correspondences (2), (4), and $W \sim w$, we may verify that the corresponding relations for the linear substitutions all reduce to identities. If therefore the order of O be proven to be $\overline{\overline{<}}$ 25920, the holohedric isomorphism between O and O_{25920} will be established. To do this, we consider the following twenty-seven sets each of 960 operators of O, those of the first set being the operators of $G \equiv G_{960}$:—*

$$\left.\begin{aligned} R_t &\equiv GW^t \\ R_{s4t} &\equiv GW^sE_3W^t \\ R_{s3t} &\equiv GW^sE_3E_2W^t \\ R_{s2t} &\equiv GW^sE_3E_2E_1W^t \\ R_{s1t} &\equiv GW^sE_3E_2E_1^2W^t \end{aligned}\right\} \quad \begin{pmatrix} t = 0, 1, 2 \\ s = 1, 2 \end{pmatrix}.$$

It will be proven in §§ 8–11 that the generators† E_1, E_2, E_3, B_1, W, and hence an arbitrary operator of O, give rise to an interchange of our 27 rows when applied as a right-hand multiplier. But the first row contains the identity. Hence the product $I.g \equiv g$, where g is an arbitrary operator of O, lies in one of the 27 rows. It follows that the order of O is at most 25920. In particular, it follows that the 27 rows form a rectangular table for O with G_{960} as first row.

We note the following formulæ derived from (8), (9), (10) :—

$$\begin{array}{lll} W^2E_2 = B_1E_2W^2, & W^2E_1 = B_1E_1W^2, & W^2B_1 = B_3B_1E_2W^2, \\ WE_2 \;= B_3W, & WE_1 \;= B_3E_2E_1W, & WB_1 \;= B_3E_2W, \\ WB_3 \;= E_2B_1W, & E_1W \;= WB_1E_1. & E_1^2W \;= WE_1^2B_1, \\ E_2W \;= WE_2B_1, & E_2W^2 = W^2B_3, & E_2E_1W = WE_2E_1, \end{array}$$

8. THEOREM.—*E_2 gives rise to the following substitution upon the* 27 *rows when applied as a right-hand multiplier :—*

$$[E_2] : \quad (R_{s10}R_{s20})(R_{s30}R_{s40})(R_{s22}R_{2s\,12})(R_{s32}R_{s42})(R_{181}R_{281})(R_{141}R_{241}),$$

where $s = 1, 2$.

---

* The notation R_t, R_{sit} for the 27 rows is that used for the corresponding rows of the rectangular table for O_{25920} as given by the writer in *Comptes Rendus*, Vol. CXXVIII. (April, 1899), pp. 873–5.

† We may drop the generator B_1 in virtue of the relation

$$B_1 \equiv W^{-1}E_1WE_1^{-1}.$$

$$R_0E_2 = R_0, \quad R_1E_2 = GWE_2 = GB_2W = GW = R_1,$$

$$R_2E_2 = GE_2B_1W^2 = R_2.$$

$$R_{s11}E_2 = GW^sE_3E_2E_1^2WE_2 = GW^sE_3E_2E_1^2B_2W = R_{s11},$$

since (11) $E_3E_2E_1^2B_2 = B_2E_3E_2E_1^2$.

$$R_{s21}E_2 = GW^sE_3E_2E_1WE_2 = GW^sE_3E_2E_1B_2W = R_{s21},$$

since (12) $E_3E_2E_1B_2 = B_1B_2E_3E_2E_1$.

$$R_{s10}E_2 = GW^sE_3E_2E_1^2E_2 = GW^sE_3E_1E_2E_1 = GW^sE_1^2E_3E_2E_1 = R_{s20}.$$

$$R_{s30}E_2 = GW^sE_3E_2.E_2 = GW^sE_3 = R_{s40}.$$

$$R_{s22}E_2 = GW^sE_3E_2E_1W^2E_2 = GW^sB_2B_3B_4E_1E_3E_2E_1^2W^2,$$

since $E_3E_2E_1E_2B_1 = B_2B_3B_4E_3E_1^2E_2E_1^2 = B_2B_3B_4E_1E_3E_2E_1^2$.

Hence, by (9), $R_{s22}E_2 = GW^{2s}E_3E_2E_1^2W^2 = R_{2s12}$,

$$R_{s32}E_2 = GW^sE_3E_2W^2E_2 = GW^sE_3E_2.E_2B_1W^2 = GW^sE_3W^2 = R_{s42}.$$

$$R_{131}E_2 = GWE_3E_2WE_2 = GWE_3E_2B_3W$$

$$= GWB_3B_4E_3E_2W = GW^2E_3E_2W = R_{231}.$$

$$R_{141}E_2 = GWE_3WE_2 = GWE_3B_3W = GW^2E_3W = R_{241}.$$

9. *When applied as a right-hand multiplier to the* 27 *rows*, E_1 *gives rise to the following substitution :—*

$$[E_1]: \quad (R_{s10}R_{s30}R_{s20})(R_{s21}R_{s31}R_{2s41})(R_{s22}R_{2s12}R_{s32}),$$

where $s = 1, 2$.

$$R_iE_1 = R_i, \quad R_{s30}E_1 = GW^sE_3E_2E_1 = R_{s20}, \quad R_{s10}E_1 = GW^sE_3E_2 = R_{s30}.$$

$$R_{s21}E_1 = GW^sE_3E_2E_1WE_1 = GW^sE_3E_2E_1B_3E_2E_1W$$

$$= GW^sE_3E_2E_1.E_2E_1W = GW^sE_3E_1^2E_2W = GW^sE_3E_2W = R_{s31},$$

using (12).

$$R_{s31}E_1 = GW^sE_3E_2WE_1 = GW^sE_3E_2B_3E_2E_1W$$

$$= GW^sB_3B_4E_3E_2.E_2E_1W = GW^{2s}E_3E_1W = GW^{2s}E_3W = R_{2s41}.$$

$$R_{s22}E_1 = GW^sE_3E_2E_1W^2E_1 = GW^sE_3E_2E_1B_1E_1W^2$$

$$= GW^sB_2B_3B_4E_3E_2E_1^2W^2 = GW^{2s}E_3E_2E_1^2W^2 = R_{2s12}.$$

$$R_{2s12}E_1 = GW^{2s}E_3E_2E_1^2B_1E_1W^2 = GW^{2s}B_1B_2B_3B_4E_3E_2W^2$$
$$= GW^sE_3E_2W^2 = R_{s32}.$$
$$R_{s11}E_1 = GW^sE_3E_2E_1^2 . B_3E_2E_1W = GW^sE_3E_2E_1^2 . E_2E_1W$$
$$= GW^sE_3E_1E_2E_1^2W = GW^sE_3E_2E_1^2W = R_{s11} \quad \text{[using (11)]}.$$
$$R_{s40}E_1 = GW^sE_3E_1 = GW^sE_1^2E_3 = GW^sE_3 = R_{s40}.$$
$$R_{s42}E_1 = GW^sE_3W^2E_1 = GW^sE_3B_1E_1W^2 = GW^sE_3W^2 = R_{s42}.$$

10. *When applied as a right-hand multiplier, W gives rise to the following substitution upon the* 27 *rows* :—

$$[W] : \quad (R_0R_1R_2)(R_{si0}R_{si1}R_{si2}) \quad (s = 1, 2;\ i = 1, 2, 3, 4).$$

11. *When applied as a right-hand multiplier, E_3 gives rise to the following substitution upon the* 27 *rows* :—

$$[E_3] : \quad (R_1R_{140})(R_2R_{240})(R_{110}R_{120})(R_{210}R_{220})(R_{211}R_{242}),$$
$$(R_{221}R_{132})(R_{112}R_{222})(R_{122}R_{141})(R_{212}R_{231})(R_{131}R_{241}).$$

We have $\qquad R_sE_3 = GW^sE_3 = R_{s40},$

$$R_{s10}E_3 = GW^sE_3E_2E_1^2E_3 = GW^sE_2E_3E_2E_1 = R_{s20},$$
$$R_{s20}E_3 = GW^sE_3E_2E_3 = GW^sE_2E_3E_2 = GW^sE_3E_2 = R_{s30},$$
$$R_{121}E_3 = GWE_3E_2E_1WE_3 = GWE_3E_2E_1W = R_{121} \quad \text{[by (10)]},$$
$$R_{221}E_3 = GW^2E_3E_2E_1WE_3 = GW(E_1^2E_2E_3E_2E_1WE_3E_2E_1W)$$
$$= GWE_3E_2E_1WE_3 . E_2E_1W = GWE_3E_2E_1W . E_2E_1W$$
$$= GWE_3E_2E_1E_2E_1W^2 = GWE_3E_1^2E_2W^2 = R_{132},$$
$$R_{211}E_3 = GW^2E_3E_2E_1^2WE_3 = GW^2E_3E_2E_1WB_1E_1E_3$$
$$= GW^2E_3E_2E_1WE_3B_1E_1^2 = R_{221}E_3B_1E_1^2 = R_{132}B_1E_1^2$$
$$= GWE_3E_2W^2B_1E_1^2 = GWE_3E_2 . B_1B_2E_2E_1^2W^2$$
$$= GWB_1B_2B_3B_4E_3E_1^2W^2 = GW^2E_3W^2 = R_{242},$$
$$R_{231}E_3 = GW^2E_3E_2WE_3 = GW^2E_3E_2E_1WE_3E_1B_1$$
$$= R_{221}E_3E_1B_1 = R_{132}E_1B_1 = GWE_3E_2W^2E_1B_1$$
$$= GWE_3E_2 . B_1E_1B_3B_1E_2W^2 = GWB_3B_4E_3E_2E_1E_2W^2$$
$$= GW^2E_3E_2E_1^2W^2 = R_{212},$$

$$\begin{aligned}
R_{131}E_3 &= GWE_3E_2WE_3 = GWE_3E_2E_1WE_3E_1B_1\\
&= GWE_3E_2E_1W.E_1B_1 = GWE_3E_2E_1.B_3E_3E_1B_3E_2W\\
&= GW^2E_3W = R_{241} \quad \text{[using (12) and (5)]},
\end{aligned}$$

$$\begin{aligned}
R_{232}E_3 &= GW^2E_3E_2W^2E_3 = GW^2E_3W^2B_3E_3 = GW^2E_3W^2E_3B_3B_4\\
&= R_{243}E_3B_3B_4 = R_{211}B_3B_4 = GW^2E_3E_2E_1^2WB_3B_4\\
&= GW^2E_3E_2E_1^2E_2B_1.B_4B_2E_1E_2E_1^2W^2\\
&= GW^2B_1B_3E_3E_2E_1^2E_2.E_1E_2E_1^2W^2 = GW^2E_3E_1^2E_2W^2\\
&= GW^2E_3E_2W^2 = R_{233},
\end{aligned}$$

$$\begin{aligned}
R_{122}E_3 &= GWE_3E_2E_1W^2E_3 = GWE_3E_2B_1W^2E_1E_3\\
&= GWE_3E_2W^2E_3E_1^2 = R_{132}E_3E_1^2 = R_{221}E_1^2\\
&= GW^2E_3E_2E_1WE_1^2 = GW^2E_3E_2E_1.E_2E_1B_2E_2E_1W\\
&= GW^2E_3B_3W = GW^2B_3B_4E_3W = GWE_3W = R_{141},
\end{aligned}$$

$$\begin{aligned}
R_{222}E_3 &= GW^2E_3E_2E_1W^2E_3 = GW^2E_3E_2.B_1W^2E_1E_3\\
&= GW^2E_3E_2W^2E_3E_1^2 = R_{232}E_3E_1^2 = R_{232}E_1^2\\
&= GW^2E_3E_2W^2E_1^2 = GW^2E_3E_2.E_1^2B_1W^2\\
&= GW^2B_1B_2B_3B_4E_3E_2E_1^2W^2 = GWE_3E_2E_1^2W^2 = R_{113},
\end{aligned}$$

$$\begin{aligned}
R_{111}E_3 &= GWE_3E_2E_1^2WE_3 = GWE_3E_2WE_3E_1B_1\\
&= R_{131}E_3E_1B_1 = R_{241}E_1B_1 = R_{121}B_1\\
&= GWE_3E_2E_1WB_1 = GWE_3E_2E_1B_3E_2W\\
&= GWE_3E_2E_1.E_2W = GWE_3E_2E_1^2W = R_{111} \quad \text{[by (12)]},
\end{aligned}$$

$$\begin{aligned}
R_{142}E_3 &= GWE_3W^2E_3 = GWE_3E_2.W^2B_3E_3 = GWE_3E_2W^2E_3B_3B_4\\
&= R_{132}E_3B_3B_4 = R_{221}B_3B_4 = GW^2E_3E_2E_1WB_3B_4\\
&= GW^2E_3E_2E_1.E_2B_1.B_4B_2E_1E_2E_1^2W^2\\
&= GW^2B_2B_4E_3E_1^2W^2 = GWE_3W^2 = R_{143}.
\end{aligned}$$

A Problem in Resonance, illustrative of the Theory of Selective Absorption of Light. By Prof. HORACE LAMB, F.R.S.

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The impact of sound-waves on a fixed spherical obstacle was discussed by Lord Rayleigh in a well known paper,[*] which also treats briefly the case where the sphere is movable, but is urged towards a fixed position by a force varying as the displacement.[†] In the present note this latter problem is studied under a more general form, it being supposed that the sphere is capable of various independent modes of free vibration; and special attention is directed to the case where there is coincidence, or approximate coincidence, between the period of the incident waves and that of one of the free modes.

The immediate acoustical importance of the question is perhaps not very great, since massive bodies are not usually set into vigorous sympathetic vibration by the direct impact of air-waves,[‡] but rather through the intermediary of resonance-boxes and sounding-boards. The problem has, however, an interest in another direction, as furnishing an analogy by which we can illustrate, without any great expenditure of analysis, the mechanical theory of selective absorption of light in a gas.

The main results of the investigation are simple, and have a high degree of generality. The "free" vibrations which can take place in the absence of extraneous sources of sound will be of the type $e^{-\nu t}\cos(\sigma t+\epsilon)$. When waves of sound beat on the sphere, there are certain values of the frequency, very nearly but not quite equal to the respective values of $\sigma/2\pi$, for which the amplitude of the scattered waves is a maximum. If the incident waves be plane, it is found that when the frequency has precisely one of these critical values the amount of energy dissipated outwards per unit time in the form of scattered waves is independent of the size of the sphere, and bears a definite ratio to the amount of energy which in the primary waves is propagated per unit time across an area equal to the square of the wave-length. Moreover, there is a rapid falling-off in the rate of dissipation as the frequency deviates from a critical value.*

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[*] *Proc. Lond. Math. Soc.*, Vol. IV., p. 253 (1892).

[†] *Ibid.*, p. 272.

[‡] *The extreme precision* of tuning that would be required militates against this.

The notations for the various solutions of the equation

$$(\nabla^2+k^2)\,\phi=0 \tag{1}$$

which are required are as explained in the author's *Hydrodynamics*, §§ 267, 309. Thus, if

$$\psi_n(\zeta)=\left(-\frac{d}{\zeta\,d\zeta}\right)^n\frac{\sin\zeta}{\zeta}=\frac{1}{1.3\ldots(2n+1)}\left\{1-\frac{\zeta^2}{2\,(2n+3)}+\ldots\right\}, \tag{2}$$

$$\Psi_n(\zeta)=\left(-\frac{d}{\zeta\,d\zeta}\right)^n\frac{\cos\zeta}{\zeta}=\frac{1.3\ldots(2n-1)}{\zeta^{2n+1}}\left\{1+\frac{\zeta^2}{2\,(2n-1)}+\ldots\right\}, \tag{3}$$

the typical solutions of (1) are $\psi_n(kr)\,r^nS_n$ and $\Psi_n(kr)\,r^nS_n$, where r is the radius vector, and S_n is a spherical surface harmonic. The former solution (alone) is finite when $r=0$. To express waves diverging from the origin, we require the combination $f_n(kr)\,r^nS_n$, where

$$f_n(\zeta)=\left(-\frac{d}{\zeta\,d\zeta}\right)^n\frac{e^{-i\zeta}}{\zeta}=\Psi_n(\zeta)-i\psi_n(\zeta). \tag{4}$$

1. Let us suppose that in the absence of a surrounding medium the sphere would have a natural mode of vibration in which the surface has the form

$$r=a+a_nS_n, \tag{5}$$

where S_n is a definite surface harmonic of order n, and a_n is a function of t. The origin is, of course, taken at the mean position of the centre of mass. Let us further suppose that the potential and kinetic energies of the sphere are determined solely by the form and by the normal motion of the surface, so that we may write

$$T=\tfrac{1}{2}A_n\left(\frac{da_n}{dt}\right)^2,\quad V=\tfrac{1}{2}C_na_n^2. \tag{6}$$

This latter assumption has, of course, only a restricted degree of generality; it holds for a fluid sphere subject to surface tension or to forces of mutual attraction, but it does not apply to a solid elastic sphere. For such purely illustrative purposes, however, as we have in view, the assumption embodied in (6) may perhaps be tolerated. The case $n=1$ is that of the vibrating rigid sphere considered by

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* The author hopes shortly to publish an investigation dealing with similar problems in an elastic solid medium. An electro-magnetic analogy is developed in *a paper recently communicated* to the Cambridge Philosophical Society (*Trans.*, *Vol. xviii., p. 348).*

Lord Rayleigh. When adverting to this case we shall replace a_1 by ξ, the displacement of the centre, and A_1 by the mass M. The harmonic S_1 is then of the form $\cos\theta$.

The equation of motion of the sphere when surrounded by a gas will be

$$A_n \frac{d^2 a_n}{dt^2} + C_n a_n = -\iint p S_n ds, \tag{7}$$

where p is the pressure and ds is an element of surface. Assuming a time-factor e^{ikct} (which it is not always necessary to show explicitly), and writing

$$\sigma_n = (C_n/A_n)^{\frac{1}{2}}, \tag{8}$$

we have

$$(\sigma_n^2 - k^2c^2) A_n a_n = -\iint p S_n ds. \tag{9}$$

The constant σ_n indicates the frequency $(\sigma_n/2\pi)$ of the oscillation when the influence of the surrounding medium is ignored.

2. Now let the velocity-potential in the gas be $\phi + \phi'$, where

$$\phi = B_n \psi_n (kr) r^n S_n, \quad \phi' = B'_n f_n (kr) r^n S_n, \tag{10}$$

ϕ being due to a certain distribution of sources at infinity, and ϕ' representing the waves reflected outwards from the sphere. If ρ be the density, the variable part of the pressure is given by

$$p = \rho \frac{d}{dt} (\phi + \phi') = ikc\rho (\phi + \phi'). \tag{11}$$

whence, substituting in (9),

$$(\sigma_n^2 - k^2c^2) A_n a_n = -ikc\rho a^n \{B_n \psi_n (ka) + B'_n f_n (ka)\} \iint S_n^2 ds. \tag{12}$$

Again, we have the kinematical relation

$$-\frac{d}{dr} (\phi + \phi') = \frac{da_n}{dt} S_n = ikca\, S_n, \tag{13}$$

whence

$$-ikca_n = B_n a^{n-1} \{ka\psi'_n (ka) + n\psi_n (ka)\} + B'_n a^{n-1} \{kaf_n (ka) + nf_n (ka)\}. \tag{14}$$

Eliminating a_n between (12) and (14), we find

$$\frac{\sigma_n^2 - k^2c^2}{k^2c^2} A_n$$

$$= -\frac{B_n \psi_n (ka) + B'_n f_n (ka)}{B_n \{ka\psi'_n (ka) + n\psi_n (ka)\} + B'_n \{kaf'_n (ka) + nf_n (ka)\}} \rho a^3 \iint S_n^2 d\varpi, \quad (1$$

where $d\varpi$, $= ds/a^2$, *is an elementary solid angle.*

3. When there are no extraneous sources, $B_n = 0$, and

$$\frac{\sigma_n^2 - k^2c^2}{k^2c^2} A_n = -\frac{f_n(ka)}{kaf_n'(ka) + nf_n(ka)} \rho a^3 \iint S_n^2 \, d\varpi. \tag{16}$$

This is an equation to determine k, and thence the character of the "free" motion of the sphere as influenced by the surrounding medium. Usually, ka will be small, so that the equation reduces to

$$\frac{\sigma_n^2 - k^2c^2}{k^2c^2} A_n = \frac{1}{n+1} \rho a^3 \iint S_n^2 \, d\varpi, \tag{17}$$

whence

$$\frac{k^2c^2}{\sigma_n^2} = \frac{1}{1+\kappa}, \tag{18}$$

if

$$\kappa = \frac{1}{n+1} \frac{\rho a^3}{A_n} \iint S_n^2 \, d\varpi. \tag{19}$$

This fixes the ratio in which the frequency of the particular mode under consideration is diminished by the inertia of the surrounding gas. In the linear oscillation of a rigid sphere ($n = 1$), we have $\kappa = \frac{2}{3}\pi\rho a^3/M$; we thus reproduce the well known result that the main effect of the fluid is to increase the inertia of the sphere by half that of the fluid displaced. To find the rate of decay of the oscillations it would be necessary to carry the approximations further. In the case $n = 1$ we find

$$\frac{kaf_1'(ka) + f_1(ka)}{f_1(ka)} = -2 + \frac{k^2a^2}{1+ika}, \tag{20}$$

and thence

$$\begin{aligned} \frac{k^2c^2}{\sigma_1^2} &= \frac{1}{1+\kappa} \left\{ 1 - \tfrac{1}{2} \frac{k^2a^2}{1+ika} \right\} \left\{ 1 - \frac{1}{2(1+\kappa)} \frac{k^2a^2}{1+ika} \right\}^{-1} \\ &= \frac{1}{1+\kappa} \left\{ 1 - \frac{\kappa}{2(1+\kappa)} k^2a^2 + \frac{\kappa}{2(1+\kappa)} ik^3a^3 + \dots \right\}. \end{aligned} \tag{21}$$

The usual method of successive approximations then leads to

$$ikc = \pm i\sigma' - \nu, \tag{22}$$

where

$$\sigma' = \frac{\sigma_1}{(1+\kappa)^{\frac{1}{2}}} \left\{ 1 - \frac{\kappa}{4(1+\kappa)^2} \frac{\sigma_1^2 a^2}{c^2} \right\}, \quad \nu = \frac{\kappa}{4(1+\kappa)} \frac{\sigma'^4 a^3}{c^3}. \tag{23}^*$$

The (realized) time-factor is now $e^{-\nu t} \cos(\sigma' t + \epsilon)$.

4. In the "forced" oscillations of order n we have, from (15) and (19),

$$\frac{B_n'}{B_n} = -\frac{\{ka\psi_n'(ka) + n\psi_n(ka)\}(\sigma_n^2 - k^2c^2) + (n+1)\kappa k^2c^2\psi_n(ka)}{\{kaf_n'(ka) + nf_n(ka)\}(\sigma_n^2 - k^2c^2) + (n+1)\kappa k^2c^2 f_n(ka)}. \tag{24}$$

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* *Cf.* Lord Rayleigh, *Theory of Sound*, § 325.

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ol. xviii.,

If ka be small, *i.e.*, if the wave-length of the extraneous sources be large in comparison with $2\pi a$, the approximate values of $\psi_n(ka)$ and $f_n(ka)$ make

$$\frac{B'_n}{B_n} = \frac{n(\sigma_n^2 - k^2c^2) + (n+1)\kappa k^2c^2}{(n+1)\{\sigma_n^2 - (1+\kappa)k^2c^2\}} \cdot \frac{(ka)^{2n+1}}{\{1.3\ldots(2n-1)\}^2(2n+1)}. \tag{25}$$

This ratio is in general of the order $(ka)^{2n+1}$, but the approximation is plainly illusory when kc is nearly equal to $\sigma_n/(1+\kappa)^{\frac{1}{2}}$, *i.e.*, when the frequency of the sources is nearly coincident with that of the free vibration. The result for a *fixed* spherical obstacle is got by putting $\sigma_n = \infty$ in (25); thus

$$\frac{B'_n}{B_n} = \frac{n}{n+1} \cdot \frac{(ka)^{2n+1}}{\{1.3\ldots(2n-1)\}^2(2n+1)}. \tag{26}$$

This requires correction when $n = 0$; it may be shown that $B'_0/B_0 = -\frac{1}{3}k^3a^3$, approximately.*

5. To examine more closely the case of approximate synchronism, let us write, in the exact formula (24),

$$f_n(ka) = \Psi_n(ka) - i\psi_n(ka). \tag{27}$$

We obtain

$$\frac{B'_n}{B_n} = -\frac{g_n(ka)}{G_n(ka) - ig_n(ka)}, \tag{28}$$

where

$$\left.\begin{aligned} g_n(ka) &= \{ka\,\psi'_n(ka) + n\psi_n(ka)\}\left(\frac{\sigma_n^2a^2}{c^2} - k^2a^2\right) + (n+1)\kappa k^2a^2\,\psi_n(ka) \\ G_n(ka) &= \{ka\,\Psi'_n(ka) + n\Psi_n(ka)\}\left(\frac{\sigma_n^2a^2}{c^2} - k^2a^2\right) + (n+1)\kappa k^2a^2\,\Psi_n(ka) \end{aligned}\right\} \tag{29}$$

The modulus of the right-hand member of (28) is never greater than unity, but it attains the value unity, and the amplitude of the scattered waves is therefore a maximum, when

$$G_n(ka) = 0, \tag{30}$$

in which case

$$B'_n = -iB_n. \tag{31}$$

If we substitute for $\Psi_n(ka)$ its value (3) in the form of a series, the little less than that given by

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* *Theory of Sound*, § 334.

$$ka = \frac{\sigma_n a}{c\,(1+\kappa)^{\frac{1}{2}}}.$$

For large values of kr we have

$$f_n(kr) = \frac{i^n e^{-ikr}}{(kr)^{n+1}}.$$

velocity-potential of the scattered waves at the origin is therefore

$$\phi' = \frac{i^n B_n'}{k^{n+1}} \frac{e^{ik(ct-r)}}{r} S_n,$$

real form, $\quad \phi' = \frac{|B_n|}{k^{n+1}} \frac{\cos k(ct-r+\epsilon)}{r} S_n.$

te of propagation of energy outwards is

$$-\iint p \frac{d\phi'}{dr} ds,$$

ver the surface of a sphere of large radius r.

$$p = \rho\, d\phi' / dt,$$

plane waves, travelling (say) in the direction of x negative, we may write

$$\phi = e^{ikr} = \Sigma\,(2n+1)(ikr)^n\,\psi_n\,(kr)\,P_n\,(\cos\theta),^{*} \tag{39}$$

where $\cos\theta = x/r$ (the origin of x being at the centre of the sphere), and P_n denotes the ordinary zonal harmonic. Hence, if we write

$$B_n = (2n+1)\,i^n k^n, \quad S_n = P_n\,(\cos\theta),$$

the scattered waves will be represented by

$$\phi = \Sigma B'_n f_n\,(kr)\,r^n P_n\,(\cos\theta), \tag{40}$$

where the ratio of B'_n to B_n is given by (24).

When k has the critical value for the component of order n, we have, by (31),
$$B'_n = -\,(2n+1)\,i^{n+1}k^n,$$

and the formula (36) becomes

$$\phi' = (-)^n\,\frac{2n+1}{k}\;\frac{\sin k\,(ct-r)}{r}\,P_n\,(\cos\theta), \tag{41}$$

corresponding to the incident wave

$$\phi = \cos k\,(x+ct). \tag{42}$$

The mean rate of dissipation of energy is now

$$2\,(2n+1)\,\pi\rho c. \tag{43}$$

The proper standard of comparison here is the energy-flux (per unit area) in the primary waves represented by (42). On the scale of our formulæ this is $\frac{1}{2}\rho k^2 c$. The ratio of (43) to this is

$$4\pi\,(2n+1)\,/\,k^2 = \frac{2n+1}{\pi}\,\lambda^2, \tag{44}$$

where λ is the wave-length.

8. In the linear oscillation $(n=1)$ the displacement ξ of the sphere is given by (14); viz., we have

$$\xi = \frac{iB_1}{kc}\,\{ka\psi'_1\,(ka)+\psi_1\,(ka)\} + \frac{iB'_1}{kc}\,\{kaf'_1\,(ka)+f_1\,(ka)\}. \tag{45}$$

When the period is adjusted to maximum dissipation this becomes

$$\xi = -\,\frac{3}{c}\,\{ka\psi'_1\,(ka)+\psi_1\,(ka)\} + \frac{3i}{c}\,\{kaf'_1\,(ka)+f_1\,(ka)\} = -\,\frac{6i}{k^3a^3c}, \tag{46}$$

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[*] See Lord Rayleigh, *l.c.*; Heine, *Kugelfunctionen*, t. I., p. 82 (1878).

approximately; or, introducing the time-factor, and taking the real part

$$\xi = \frac{6}{k^3a^3c} \sin kct. \tag{47}$$

The amplitude of vibration of the air particles in the original wave (42) is $1/c$ on the present scale; the amplitude of the sphere exceeds this in the ratio $6/k^3a^3$. Again, the energy scattered in any time is, by (44), equal to ·955 of that which in the primary waves crosses in the same time a square area whose side is equal to the wave-length. If the sphere were *fixed*, the expression corresponding to (44) would be

$$\tfrac{7}{9} (ka)^4 \pi a^2.\text{*} \tag{48}$$

The ratio of (44) to this (when $n = 1$) is $\frac{108}{7} (ka)^{-6}$.

As stated in the introduction, these results are interesting chiefly for the optical analogy. They indicate that, on any mechanical theory of selective absorption, the dissipation in the case of synchronism will be so great that an assemblage of molecules, such as is supposed to constitute a gas,† would be absolutely impenetrable to waves of the critical length.

9. The wave-length of maximum dissipation is very sharply defined, and the more sharply the greater the value of n. If we put

$$ka = (1+\epsilon) z, \tag{50}$$

where z is the small root of

$$G_n (z) = 0 \tag{51}$$

and ϵ is a small fraction, it will be found that

$$g_n (ka) = \frac{\kappa z^2}{1.3 \dots (2n-1)}, \quad G_n (ka) = \frac{2(n+1)\, 1.3 \dots (2n-1)}{z^{2n-1}} (1+\kappa) \epsilon, \tag{52}$$

approximately. The formula (28) then shows that the dissipation is less than the maximum in the ratio

$$\left[1 + \frac{4(n+1)^2 \{1.3 \dots (2n-1)\}^4}{z^{4n+2}} \left(\frac{1+\kappa}{\kappa}\right)^2 \epsilon^2\right]^{-1}. \tag{53}$$

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* This is easily deduced from (38) with the values of B_1' and B_0' given in (26) and the line following. The result is given by Lord Rayleigh, *Phil. Mag.*, April, 1899.

† *Viz.*, the distances between neighbouring molecules are large compared with the diameters, whilst a very large number of molecules is contained in a volume λ^3.

This ratio has the value $\frac{1}{2}$ when

$$\epsilon = \pm \frac{z^{2n+1}}{2(n+1)\{1.3\dots(2n-1)\}^2}\,\frac{\kappa}{1+\kappa}. \tag{54}$$

For example, when $n = 1$, the dissipation sinks to one-half the maximum when the wave-length of the incident vibration deviates from the critical value by the fraction

$$\frac{\kappa z^3}{4(1+\kappa)}$$

of itself. For $n = 2$, which may be taken as corresponding roughly to the case of a tuning-fork (without resonance-box or sounding-board), the corresponding fraction is

$$\frac{\kappa z^5}{54(1+\kappa)}.$$

It is to be noticed that not only is z $(= 2\pi a/\lambda)$ small, by hypothesis, but that κ will in any acoustical application be a very minute fraction.

10. Regarded as an illustration of optical conditions, the notion of a molecule attracted towards a fixed position in space is of course somewhat crude. Lord Kelvin has employed on various occasions the conception of a rigid spherical shell enclosing other concentric shells which are attached to it and to one another by massless springs. It is easy to adapt our analysis to such a case. The relative configuration of any dynamical system enclosed by the outer shell may be specified by means of coordinates $q_1, q_2, \dots, q_m$; and there is no loss of generality in assuming these latter coordinates to be so chosen that the expressions for the kinetic and potential energies of the shell together with the included system reduce to the forms

$$2T = M\dot{\xi}^2 + 2(\alpha_1\dot{q}_1 + \alpha_2\dot{q}_2 + \dots + \alpha_m\dot{q}_m)\,\dot{\xi} + a_1\dot{q}_1^2 + a_2\dot{q}_2^2 + \dots + a_m\dot{q}_m^2 \tag{55}$$

and

$$2V = c_1 q_1^2 + c_2 q_2^2 + \dots + c_m q_m^2, \tag{56}$$

respectively, where ξ refers to the outer shell. Lagrange's method gives m equations of the type

$$a_r\ddot{q}_r + \alpha_r\ddot{\xi} + c_r q_r = 0, \tag{57}$$

together with

$$M\ddot{\xi} + \Sigma_r\alpha_r\ddot{q}_r = -\iint p\cos\theta\, dS. \tag{58}$$

Introducing the time-factor e^{ikct}, we find

$$q_r = -\frac{k^2c^2a_r}{k^2c^2a_r - c_r}\xi, \tag{59}$$

and thence $$\left\{-k^2c^2M + k^4c^4\Sigma_r \frac{a_r^2}{k^2c^2a_r - c_r}\right\}\xi = -\iint p\cos\theta\, ds. \tag{60}$$

This corresponds to (9), with $n = 1$, and the work can then be continued as before. The result will be that the factor $\sigma_1^2 - k^2c^2$, where it appears in (16) or in (24) (with $n = 1$), must be replaced by

$$-k^2c^2 + \frac{k^4c^4}{M}\Sigma_r \frac{a_r^2}{k^2c^2a_r - c_r}. \tag{61}$$

It is easily verified that, when waves of given period impinge on the system, there are certain critical values of the period for which the dissipation is a maximum; moreover, that in the case of plane incident waves the ratio of the maximum dissipation to the primary energy-flux has the same value as before, viz.,

$$\frac{3}{\pi}\lambda^2. \tag{62}$$

A (mathematically) singular case arises when

$$kc = (c_r/a_r)^{\frac{1}{2}},$$

i.e., when the frequency of the incident waves coincides with what would be the period of the free vibration corresponding to the coordinate q_r, if the outer shell were fixed. It appears from the preceding equations that ξ will vanish, together with all the coordinates $q_1, q_2, \ldots, q_m$, with the exception of q_r. The ratio of B_1' to B_1 is determined by the equation

$$\frac{d}{dr}(\phi + \phi') = 0, \tag{63}$$

for $r = a$; and the value of q_r then follows from the equation (58), which reduces to

$$q_r = \frac{1}{k^2c^2a_r}\iint p\cos\theta\, ds. \tag{64}$$

It is otherwise evident that, under the circumstances supposed, a steady internal vibration is possible, whose reaction on the enclosing shell shall just balance the pressure of the air-waves upon it.

Thursday, February 8th, 1900.

Prof. ELLIOTT, F.R.S., Vice-President, and subsequently Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President, in the Chair.

Twelve members present.

Prof. Elliott announced that the Council had passed the following resolution, viz.:—

> "That, the objects of The London Mathematical Society requiring that it shall consist of more than 250 members, it is resolved that the number of its members may be increased by further elections to 350."

He also mentioned that the Registrary of the University of Cambridge, by direction of the Reception Committee, had forwarded to the Society a bronze copy of the medal presented to Sir G. G. Stokes by the University, on the occasion of his jubilee. The medal was exhibited to the members present.

Prof. Love gave a sketch of a paper by Mr. J. H. Michell, entitled "Some Elementary Distributions of Stress in Three Dimensions." Dr. Larmor joined in a discussion of the paper.

Major MacMahon spoke on "Combinatorial Analysis, the foundation of a new Theory."

The following papers, in the absence of their authors, were taken as read:—

A Formula in the Theory of the Theta-Functions, by Prof. A. C. Dixon.

The Canonical Reduction of a Pair of Bilinear Forms, and Reduction of a Generalized Linear Substitution to a Canonical Form, with a Dynamical Application, by Mr. T. J. I'A. Bromwich.

The following presents were made to the Library:—

"Educational Times," February, 1900.

"Indian Engineering," Vol. XXVI., Nos. 26, 27, 1899; Vol. XXVII., Nos. 1, 2, 1900; Dec. 23–Jan. 13.

Kneser, A.—"*Lehrbuch der Variationsrechnung,*" 8vo; Braunschweig, 1900.

Forsyth, A. R.—"Theory of Differential Equations," Pt. 2, Vols. II., III., 8vo; Cambridge, 1900.

"The Astronomical and Astrophysical Society of America," First Meeting, 1899 (from "Science," Vol. X., No. 25).

Schreibner, W.—"Zur Theorie des Legendre-Jacobi'schen Symbols $\left(\frac{n}{m}\right)$," 8vo; Leipzig, 1900.

Ostwald, W.—"Dampfdrucke Ternärer Gemische," 8vo; Leipzig, 1900.

The following exchanges were received:—

"Periodico di Matematica," Anno XV., Fasc. 3, 4; Livorno, 1900.

"Proceedings of the Royal Society," Vol. LXV., Nos. 422, 423, 1900.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXIII., St. 12, Bd. XXIV., St. 1; Leipzig, 1899-1900.

"Bulletin de la Société Mathématique de France," Tome XXVII., Fasc. 4; Paris.

"Bulletin of the American Mathematical Society," Series 2, Vol. VI., No. 4; New York, 1900.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIII., No 6; Coimbra, 1899.

"Monatshefte für Mathematik und Physik," Jahrgang XI., No. 1; Wien, 1900.

"Bulletin des Sciences Mathématiques," Tome XXIII., Nov.; Paris, 1899.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. V., Fasc. 8-12; Napoli, 1899.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. IX., Fasc. 1; Roma, 1900.

"Revue Semestrielle des Publications Mathématiques," Tome VIII., Partie 1, Av.-Oct., 1899.

"Journal of the Institute of Actuaries," Vol. XXXV., Pt. 2; January, 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 39-53; Oct. 19-Dec. 21, 1899.

"Proceedings of the Cambridge Philosophical Society," Vol. X., Pt. 4; 1900.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXVIII., Heft 3, Jahrgang, 1897; Berlin, 1900.

"Annals of Mathematics," Series 2, Vol. I., No. 2; Harvard, 1900.

Some Elementary Distributions of Stress in Three Dimensions. *By* J. H. Michell.

Received January 21st, 1900. Communicated February 8th, 1900.

A force is applied to an isotropic elastic solid, either at an internal point, or on a plane boundary, or at the vertex of a cone, and it is required to find the distribution of stress. If we make the assumption that the stresses vary inversely as the square of the distance from the point of application of the force, we arrive at a series of elementary solutions of which those suitable for the present problems are here communicated. These elementary solutions must of course include those at the base of the classical works of Thomson and Tait,[*] Boussinesq,[†] and Cerruti on the infinite solid and the solid with an infinite plane boundary.

The General Equations.

1. Let u', v', w' be the displacements along rectangular axes. The equations of equilibrium are

$$(\lambda+\mu)\frac{d\Delta}{dx}+\mu\nabla^2 u'=0,$$

$$(\lambda+\mu)\frac{d\Delta}{dy}+\mu\nabla^2 v'=0,$$

$$(\lambda+\mu)\frac{d\Delta}{dz}+\mu\nabla^2 w'=0.$$

Multiplying by x, y, z respectively, and adding, we get

$$(\lambda+\mu)\,r\frac{d\Delta}{dr}+\mu\nabla^2(xu'+yv'+zw')-2\mu\Delta=0,$$

or

$$(\lambda+\mu)\,r\frac{d\Delta}{dr}-2\mu\Delta+\mu\nabla^2 rw=0,$$

where u, v, w are the spherical polar displacements.

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[*] *Natural Philosophy*, § 730.

[†] *Applications des Potentiels.*

For the rotations ϖ_1', ϖ_2', ϖ_3', we have

$$\nabla^2 \varpi_1' = 0,$$
$$\nabla^2 \varpi_2' = 0,$$
$$\nabla^2 \varpi_3' = 0.$$

Multiplying by x, y, z respectively, and adding, we get

$$\nabla^2 (x\varpi_1' + y\varpi_2' + z\varpi_3') = 0,$$

or

$$\nabla^2 r\varpi_3 = 0,$$

ϖ_1, ϖ_2, ϖ_3 being the spherical-polar rotations.

We have therefore the three intrinsic equations of equilibrium

$$\left.\begin{aligned} \nabla^2\Delta &= 0 \\ \mu\nabla^2 rw + (\lambda+\mu)\, r\frac{d\Delta}{dr} - 2\mu\Delta &= 0 \\ \nabla^2 r\varpi_3 &= 0 \end{aligned}\right\}. \qquad (1)$$

These are not, in general, *sufficient* equations of equilibrium. Let a solution of (1) give

$$(\lambda+\mu)\frac{d\Delta}{dx} + \mu\nabla^2 u' = -X',$$
$$(\lambda+\mu)\frac{d\Delta}{dy} + \mu\nabla^2 v' = -Y',$$
$$(\lambda+\mu)\frac{d\Delta}{dz} + \mu\nabla^2 w' = -Z'.$$

Then, from the process employed in obtaining equations (1), we must have

$$xX' + yY' + zZ' = 0,$$
$$x(Z_y' - Y_z') + y(X_z' - Z_x') + z(Y_x' - X_y') = 0,$$
$$X_x' + Y_y' + Z_z' = 0.$$

Introducing the polar components of force X, Y, Z, these equations are known to be equivalent to

$$Z = 0,$$
$$\frac{d}{d\theta}(Y\sin\theta) - \frac{dX}{d\phi} = 0,$$
$$\frac{d}{d\theta}(X\sin\theta) + \frac{dY}{d\phi} = 0.$$

It follows that, if the solution of (1) is regular throughout the solid angle, we must have $X = Y = 0$, that is, the solution satisfies the equations of equilibrium with no volume force. In this statement no attention is paid to an irregularity at $\theta = 0$ or $\theta = \pi$, which disappears if the polar axis is displaced. The application of (1) to complete spheres and spherical shells must be left to another occasion. At present we consider those solutions for which the displacements vary inversely as the distance from the origin, and the stresses, consequently, inversely as the square of the distance. In this case, after having found the general solution of (1) of the form required, we substitute in the equations*

$$X = 0 = (\lambda+2\mu)\sin\theta\frac{d\Delta}{d\theta} - 2\mu\frac{d\varpi_3}{d\phi} + 2\mu\sin\theta\frac{d}{dr}(r\varpi_2),$$

$$Y = 0 = (\lambda+2\mu)\operatorname{cosec}\theta\frac{d\Delta}{d\phi} - 2\mu\frac{d}{dr}(r\varpi_1) + 2\mu\frac{d\varpi_3}{d\theta},$$

which, remembering that ur, vr are independent of r, become

$$\left.\begin{aligned} 0 &= (\lambda+2\mu)\,r\sin\theta\frac{d\Delta}{d\theta} - 2\mu r\frac{d\varpi_3}{d\phi} + \mu\sin\theta\frac{dw}{d\theta} \\ 0 &= (\lambda+2\mu)\,r\frac{d\Delta}{d\phi} + 2\mu r\sin\theta\frac{d\varpi_3}{d\theta} + \mu\frac{dw}{d\phi} \end{aligned}\right\}. \tag{2}$$

The General Solution.

2. When $r\Delta$, $r\varpi_3$, w vary inversely as r, the equations (1) become

$$\frac{d^2\Delta}{d\theta^2} + \cot\theta\frac{d\Delta}{d\theta} + \operatorname{cosec}^2\theta\frac{d^2\Delta}{d\phi^2} + 2\Delta = 0,$$

$$\mu\left(\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta} + \operatorname{cosec}^2\theta\frac{d^2w}{d\phi^2}\right) - 2(\lambda+2\mu)\Delta = 0,$$

$$\frac{d^2\varpi_3}{d\theta^2} + \cot\theta\frac{d\varpi_3}{d\theta} + \operatorname{cosec}^2\theta\frac{d^2\varpi_3}{d\phi^2} = 0.$$

Assuming that u, w vary as $\cos n\phi$, and that v varies as $\sin n\phi$, these equations become

$$\frac{d^2\Delta}{d\theta^2} + \cot\theta\frac{d\Delta}{d\theta} + (2 - n^2\operatorname{cosec}^2\theta)\,\Delta = 0, \tag{3}$$

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* Love, *Elasticity*, I., § 125.

$$\mu\left(\frac{d^2w}{d\theta^2}+\cot\theta\frac{dw}{d\theta}-n^2\operatorname{cosec}^2\theta\, w\right)-2(\lambda+2\mu)\,r\Delta=0, \tag{4}$$

$$\frac{d^2\varpi_3}{d\theta^2}+\cot\theta\frac{d\varpi_3}{d\theta}-n^2\operatorname{cosec}^2\theta\,\varpi_3=0. \tag{5}$$

The general solution of (3) is

$$r^2\Delta/\cos n\phi=A\tan^n\frac{\theta}{2}(n+\cos\theta)+B\cot^n\frac{\theta}{2}(n-\cos\theta), \tag{6}$$

that of (4) is

$$rw/\cos n\phi=-\frac{\lambda+2\mu}{\mu}\;\frac{r^3\Delta}{\cos n\phi}+C\tan^n\frac{\theta}{2}+D\cot^n\frac{\theta}{2}, \tag{7}$$

and that of (5) is

$$2r^3\varpi_3/\sin n\phi=E\tan^n\frac{\theta}{2}+F\cot^n\frac{\theta}{2}. \tag{8}$$

If we substitute these solutions in equations (2), we have

$$\sin\theta\frac{d}{d\theta}\left(C\tan^n\frac{\theta}{2}+D\cot^n\frac{\theta}{2}\right)-n\left(E\tan^n\frac{\theta}{2}+F\cot^n\frac{\theta}{2}\right)=0,$$

$$-n\left(C\tan^n\frac{\theta}{2}+D\cot^n\frac{\theta}{2}\right)+\sin\theta\frac{d}{d\theta}\left(E\tan^n\frac{\theta}{2}+F\cot^n\frac{\theta}{2}\right)=0,$$

which are both satisfied if

$$C=E,$$

$$D=-F.$$

The general solution of the assumed form is therefore

$$r^3\Delta/\cos n\phi=A\tan^n\frac{\theta}{2}(n+\cos\theta)+B\cot^n\frac{\theta}{2}(n-\cos\theta),$$

$$rw/\cos n\phi=-\frac{\lambda+2\mu}{\mu}\;\frac{r^3\Delta}{\cos n\phi}+C\tan^n\frac{\theta}{2}+D\cot^n\frac{\theta}{2},$$

$$2r^2\varpi_3/\sin n\phi=C\tan^n\frac{\theta}{2}-D\cot^n\frac{\theta}{2}.$$

To find the displacements u, v, we have

$$\frac{d}{d\theta}(ur\sin\theta)+\frac{d}{d\phi}(vr)=r^2\sin\theta\Delta-wr\sin\theta,$$

$$\frac{d}{d\theta}(vr\sin\theta)-\frac{d}{d\phi}(ur)=2\varpi_3r^3\sin\theta,$$

and it is readily shown that the solution of these is

$$ur\sin\theta/\cos n\phi = -\frac{\lambda+3\mu}{2\mu}\sin\theta\frac{d}{d\theta}\left(\frac{r^2\Delta}{\cos n\phi}\right)$$

$$+\left(C\tan^n\frac{\theta}{2}+D\cot^n\frac{\theta}{2}\right)\cos\theta+G\tan^n\frac{\theta}{2}+H\cot^n\frac{\theta}{2},$$

$$vr\sin\theta/\sin n\phi = n\frac{\lambda+3\mu}{2\mu}\frac{r^2\Delta}{\cos n\phi}-\left(C\tan^n\frac{\theta}{2}-D\cot^n\frac{\theta}{2}\right)\cos\theta$$

$$-G\tan^n\frac{\theta}{2}+H\cot^n\frac{\theta}{2}.$$

In the particular solutions we are about to consider $n = 0$ or 1. In these two cases some of the solutions change form, but those which we use are at once derivable from the general solutions.

Force at a Point of an Infinite Solid.

3. Take the polar axis in the direction of the applied force F at the origin. In the general solution we must put $n = 0$, and there is the further condition of regularity throughout the solid angle. The required solution is readily found to be

$$u = -A\frac{\lambda+3\mu}{2\mu}\frac{\sin\theta}{r},$$

$$v = 0,$$

$$w = A\frac{\lambda+2\mu}{\mu}\frac{\cos\theta}{r},$$

where $$A = \frac{1}{\lambda+2\mu}\frac{F}{4\pi}.$$

The curves of displacement are determined by

$$\tan\psi = \frac{u}{w} = -\frac{\lambda+3\mu}{2(\lambda+2\mu)}\tan\theta,$$

where ψ is the angle between the tangent and the radius.

For incompressible material

$$\tan\psi = -\tfrac{1}{2}\tan\theta,$$

which gives $$r\sin^2\theta = C$$

as the *curves of displacement.*

For Poisson's isotropic material $\lambda = \mu$, and

$$\tan\psi = -\tfrac{2}{3}\tan\theta,$$

which gives
$$r^2 \sin^3\theta = C$$
as the curves of displacement.

The strains are
$$e = A\frac{\lambda+\mu}{2\mu}\frac{\cos\theta}{r^2},$$
$$f = A\frac{\lambda+\mu}{2\mu}\frac{\cos\theta}{r^2},$$
$$g = -A\frac{\lambda+2\mu}{\mu}\frac{\cos\theta}{r^2},$$
$$\Delta = -A\frac{\cos\theta}{r^2},$$
$$a = c = 0,$$
$$b = A\frac{\sin\theta}{r^2};$$

and the stresses
$$P = Q = \mu A\frac{\cos\theta}{r^2},$$
$$R = -(3\lambda+4\mu)A\frac{\cos\theta}{r^2},$$
$$S = U = 0,$$
$$T = \mu A\frac{\sin\theta}{r^2}.$$

For an incompressible solid $\mu/\lambda = 0$, and the solution reduces to

$$P = Q = S = T = U = 0,$$
$$R = -A'\frac{\cos\theta}{r^2} = -\frac{3}{4\pi}F\frac{\cos\theta}{r^2}.$$

In an incompressible solid, we have, therefore, a purely radial distribution of stress, and, as in the case of plane stress,* the same solution applies to a force applied in any direction at the vertex of a cone of any shape, including as particular cases the wedge and the infinite plane.

In the general case the directions of the principal axes of stress in the meridian planes are given by

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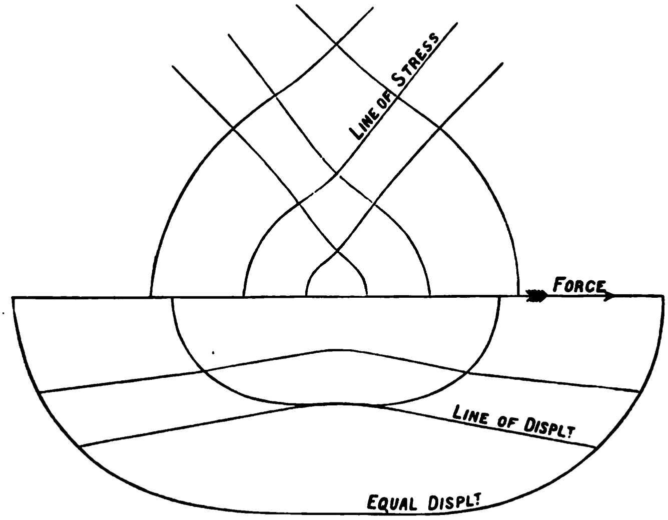
* "*Elementary* Distributions of Plane Stress," *infra*, pp. 35 *et seq*.

$$\tan 2\psi = \frac{2T}{R-P} = -\frac{2\mu}{3\lambda+5\mu}\tan\theta.$$

For Poisson's material $\tan 2\psi = -\frac{1}{4}\tan\theta$;

by means of which we can trace the lines of stress as in the accompanying figure. The lines of displacement and the curves of equal displacement, for the same case, are indicated in the same figure.



The approximation to pure radial stress in the neighbourhood of $\theta = 0$, the line of the applied force, is an important feature from the practical standpoint.

Force at the Vertex of a Cone and along the Axis.

4. Let the force F be applied at the vertex of a right circular cone in the direction of its axis. Taking the axis of the cone as the polar axis, consider the elementary solution

$$u = -\frac{B}{r}\tan\frac{\theta}{2},$$

$$v = 0,$$

$$w = \frac{B}{r}.$$

The corresponding strains are

$$e = \tfrac{1}{2}\frac{B}{r^2}\sec^2\frac{\theta}{2}\cos\theta,$$

$$f = \tfrac{1}{2}\frac{B}{r^2}\sec^2\frac{\theta}{2},$$

$$g = -\frac{B}{r^2},$$

$$\Delta = 0,$$

$$a = c = 0,$$

$$b = \frac{2B}{r^2}\tan\frac{\theta}{2};$$

and the stresses

$$P = \mu\frac{B}{r^2}\sec^2\frac{\theta}{2}\cos\theta,$$

$$Q = \mu\frac{B}{r^2}\sec^2\frac{\theta}{2},$$

$$R = -2\mu\frac{B}{r^2}$$

$$S = U = 0,$$

$$T = 2\mu\frac{B}{r^2}\tan\frac{\theta}{2}.$$

This solution is readily shown to be identical with Boussinesq's *second type of simple solutions.*[*]

Combine this solution with that of § 3, making

$$B = -A\cos^2\frac{a}{2}.$$

We thus get the distribution

$$P = \tfrac{1}{2}\mu\frac{A}{r^2}(\cos\theta - \cos a)\cos\theta\sec^2\frac{\theta}{2},$$

$$Q = \tfrac{1}{2}\mu\frac{A}{r^2}(\cos\theta - \cos a - \sin^2\theta)\sec^2\frac{\theta}{2},$$

$$R = \frac{A}{r^2}\left\{-(3\lambda+4\mu)\cos\theta + \mu(1+\cos a)\right\},$$

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[*] Love, *Elasticity*, § 161.

$$S = U = 0,$$

$$T = \tfrac{1}{2}\mu \frac{A}{r^2} (\cos\theta - \cos\alpha) \sin\theta \sec^2 \frac{\theta}{2}.$$

Since $P = T = 0$, when $\theta = \alpha$, this is the required solution for a cone of semivertical angle α.

The equality $P \sin\theta = T \cos\theta$ shows that the stress across a normal section is directed radially from the vertex of the cone.

By considering the stress across the spherical section r of the cone, we find for the magnitude of the applied force

$$F = 2\pi \int_0^\alpha T \sin^2\theta \, d\theta - 2\pi \int_0^\alpha R \cos\theta \sin\theta \, d\theta$$
$$= 2\pi A \left\{\lambda\,(1 - \cos^2\alpha) + \mu\,(1 - \cos\alpha)(1 + \cos^2\alpha)\right\}.$$

For Poisson's material

$$F = 2\pi\mu A\,(2 + \cos\alpha + 2\cos^2\alpha)(1 - \cos\alpha).$$

The preponderance of the radial stress is worth noting. For many purposes it would be sufficient to neglect P, Q, T and take simply

$$R = -A\cos\theta / r^2,$$

as if the material were incompressible.

If we put $\alpha = \frac{\pi}{2}$, we have Boussinesq's elementary solution for a normal force at a point of an infinite plane boundary. This solution in polar coordinates is

$$P = \tfrac{1}{2}\mu \frac{A}{r^2} \cos^2\theta \sec^2 \frac{\theta}{2},$$

$$Q = \tfrac{1}{2}\mu \frac{A}{r^2} (\cos\theta - \sin^2\theta) \sec^2 \frac{\theta}{2},$$

$$R = \frac{A}{r^2} \left\{-(3\lambda + 4\mu)\cos\theta + \mu\right\},$$

$$T = \tfrac{1}{2}\mu \frac{A}{r^2} \sin\theta \cos\theta \sec^2 \frac{\theta}{2},$$

where $$F = 2\pi A\,(\lambda + \mu).$$

Force at Right Angles to the Axis of the Cone.

5. Taking the polar axis along the axis of the cone, and the force in the direction $\left(\theta = \frac{\pi}{2},\ \phi = 0\right)$, we consider three elementary solutions.

(i.) This is merely the solution of § 3 with the force in the new direction.

We have
$$u = -\frac{C}{r}\cos\theta\cos\phi,$$
$$v = \frac{C}{r}\sin\phi,$$
$$w = -2\frac{\lambda+2\mu}{\lambda+3\mu}\,\frac{C}{r}\sin\theta\cos\phi.$$

These give
$$P = -\frac{2\mu^2}{\lambda+3\mu}\,\frac{C}{r^2}\sin\theta\cos\phi,$$
$$Q = -\frac{2\mu^2}{\lambda+3\mu}\,\frac{C}{r^2}\sin\theta\cos\phi,$$
$$R = \frac{2\mu\,(3\lambda+4\mu)}{\lambda+3\mu}\,\frac{C}{r^2}\sin\theta\cos\phi,$$
$$S = -\frac{2\mu^2}{\lambda+3\mu}\,\frac{C}{r^2}\sin\phi,$$
$$T = \frac{2\mu^2}{\lambda+3\mu}\,\frac{C}{r^2}\cos\theta\cos\phi,$$
$$U = 0.$$

(ii.) This solution is
$$u = \frac{D}{r}\cos\phi,$$
$$v = -\frac{D}{r}\sin\phi,$$
$$w = \frac{D}{r}\tan\frac{\theta}{2}\cos\phi.$$

It gives
$$e = \frac{D}{r^2}\tan\frac{\theta}{2}\cos\phi,$$
$$f = 0,$$
$$g = -\frac{D}{r^2}\tan\frac{\theta}{2}\cos\phi,$$

$$\Delta = 0,$$

$$a = \frac{D}{r^3}\left(2-\tfrac{1}{2}\sec^2\frac{\theta}{2}\right)\sin\phi,$$

$$b = -\frac{D}{r^3}\left(2-\tfrac{1}{2}\sec^2\frac{\theta}{2}\right)\cos\phi,$$

$$c = -\frac{D}{r^3}\tan\frac{\theta}{2}\sin\phi;$$

and therefore

$$P = 2\mu\frac{D}{r^3}\tan\frac{\theta}{2}\cos\phi,$$

$$Q = 0,$$

$$R = -2\mu\frac{D}{r^3}\tan\frac{\theta}{2}\cos\phi,$$

$$S = \mu\frac{D}{r^3}\left(2-\tfrac{1}{2}\sec^2\frac{\theta}{2}\right)\sin\phi,$$

$$T = -\mu\frac{D}{r^3}\left(2-\tfrac{1}{2}\sec^2\frac{\theta}{2}\right)\cos\phi,$$

$$U = -\mu\frac{D}{r^3}\tan\frac{\theta}{2}\sin\phi.$$

(iii.) This solution is

$$u = -\frac{E}{r}\sec^2\frac{\theta}{2}\cos\phi,$$

$$v = \frac{E}{r}\sec^2\frac{\theta}{2}\sin\phi,$$

$$w = 0.$$

It gives

$$e = -\frac{E}{r^2}\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\cos\phi,$$

$$f = \frac{E}{r^2}\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\cos\phi,$$

$$g = 0,$$

$$\Delta = 0,$$

$$a = -\frac{2E}{r^2}\sec^2\frac{\theta}{2}\sin\phi,$$

$$b = \frac{2E}{r^2}\sec^2\frac{\theta}{2}\cos\phi,$$

$$c = \frac{2E}{r^2}\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}\sin\phi;$$

and therefore
$$P = -2\mu \frac{E}{r^2} \tan\frac{\theta}{2} \sec^2\frac{\theta}{2} \cos\phi,$$
$$Q = \quad 2\mu \frac{E}{r^2} \tan\frac{\theta}{2} \sec^2\frac{\theta}{2} \cos\phi,$$
$$R = 0,$$
$$S = -2\mu \frac{E}{r^2} \sec^2\frac{\theta}{2} \sin\phi,$$
$$T = \quad 2\mu \frac{E}{r^2} \sec^2\frac{\theta}{2} \cos\phi,$$
$$U = \quad 2\mu \frac{E}{r^2} \tan\frac{\theta}{2} \sec^2\frac{\theta}{2} \sin\phi.$$

We can now combine these solutions so as to make
$$P = T = U = 0,$$
when
$$\theta = a.$$

The result is
$$u = \frac{K}{r}\left(-\frac{\lambda+3\mu}{2\mu}\cos\theta + 2\cos^2\frac{a}{2} - \sec^2\frac{\theta}{2}\cos^4\frac{a}{2}\right)\cos\phi,$$
$$v = \frac{K}{r}\left(\frac{\lambda+3\mu}{2\mu} - 2\cos^2\frac{a}{2} + \sec^2\frac{\theta}{2}\cos^4\frac{a}{2}\right)\sin\phi,$$
$$w = \frac{K}{r}\left(-\frac{\lambda+2\mu}{\mu}\sin\theta + 2\tan\frac{\theta}{2}\cos^2\frac{a}{2}\right)\cos\phi,$$
$$P = -\frac{\mu}{2}\frac{K}{r^2}\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}(\cos\theta - \cos a)^2\cos\phi,$$
$$Q = -\frac{\mu}{2}\frac{K}{r^2}\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}(\cos a + \cos\theta + 2)(\cos\theta - \cos a)\cos\phi,$$
$$R = \mu\frac{K}{r^2}\left\{\frac{3\lambda+2\mu}{\mu}\sin\theta + 2\tan\frac{\theta}{2}(\cos\theta - \cos a)\right\}\cos\phi,$$
$$S = \quad \frac{\mu}{2}\frac{K}{r^2}\left\{\cos a(1-\cos a) - \tan^2\frac{\theta}{2}(1+\cos a)(2+\cos a)\right\}\sin\phi,$$
$$T = -\frac{\mu}{2}\frac{K}{r^2}\left\{\cos a + (2+\cos a)\tan^2\frac{\theta}{2}\right\}(\cos\theta - \cos a)\cos\phi,$$
$$U = -\frac{\mu}{2}\frac{K}{r^2}\tan\frac{\theta}{2}\sec^2\frac{\theta}{2}(1+\cos a)(\cos\theta - \cos a)\sin\phi.$$

The magnitude of the applied force F' is found by integrating over a spherical section r. We find

$$F' = -\pi K\{\lambda(2+\cos\alpha)+2\mu\}(1-\cos\alpha)^2,$$

If $\lambda=\mu$, $\quad F' = -\pi\mu K(4+\cos\alpha)(1-\cos\alpha)^2.$

Combining the result of this and the preceding section, we obtain the distribution of stress due to a force applied in any direction at the vertex of the cone. If we put $\alpha = \frac{\pi}{2}$, we have the solution for a tangential force on a plane boundary. It is of practical interest to find the limits within which the force must lie that there may be no radial tension. If we make $R=0$, when $\phi=\pi$, $\theta=\alpha$, we have, supposing $\lambda=\mu$,

$$-5\sin\alpha K+(1-6\cos\alpha)A = 0,$$

and the limiting direction of the force is given by

$$\tan\psi = \frac{F'}{F} = \frac{-K(4+\cos\alpha)(1-\cos\alpha)^2}{2A(2+\cos\alpha+2\cos^2\alpha)(1-\cos\alpha)}$$

$$= \frac{(6\cos\alpha-1)(4+\cos\alpha)(1-\cos\alpha)}{10\sin\alpha(2+\cos\alpha+2\cos^2\alpha)}.$$

If α is small, $\quad \tan\psi = \tfrac{1}{4}\alpha,$

so that the force must lie within the middle quarter of the vertical angle, corresponding to the law of the middle third in plane stress.

The same method can clearly be applied to the hollow cone by adding solutions not regular at $\theta=0$.

Elementary Distributions of Plane Stress. By J. H. Michell.

Received January 7th, 1900. Communicated January 11th, 1900.

In a recent paper[*] I have discussed the fundamental equations of plane stress. I now propose to consider some elementary solutions by means of which various interesting problems can be solved in a very simple form. The solutions given apply to solids indefinitely extended in one direction (infinite cylinders) as well as to thin plates, an alteration of the moduluses alone being required to pass from the one case to the other.[*] The method employed was first suggested by a consideration of the transmission of stress from a load on an isotropic wall. A search through the accessible literature resulted in finding that Flamant[†] and Boussinesq[‡] had discussed a problem equivalent to this and had recognized the extreme simplicity of the result; but I cannot find that they have gone further in the direction of the present paper. The same simple distribution of stress had, however, been previously applied by Hertz[§] to give a remarkably simple solution for the stress in a circular cylinder, as I recognized after rediscovering his results in a more geometrical form.

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[*] "On the Direct Determination of Stress, &c.," Lond. Math. Soc. Proc., Vol. xxxi., p. 100.

Force at a Point of an Æolotropic Plate.

1. Taking polar coordinates having their origin at the point of application of the force, the resolution-equations of stress are[||]

$$\frac{dP}{dr}+\frac{1}{r}\frac{dU}{d\theta}+\frac{P-Q}{r}=0,$$

$$\frac{dU}{dr}+\frac{1}{r}\frac{dQ}{d\theta}+\frac{2U}{r}=0.$$

Making the almost obviously correct assumption that the stresses vary inversely as the distance from the point of application of the force, the equations reduce to

$$\frac{dU}{d\theta}-Q=0,$$

$$\frac{dQ}{d\theta}+U=0,$$

of which the general solution is

$$\left.\begin{aligned} Q &= A\cos(\theta+\alpha)/r \\ U &= A\sin(\theta+\alpha)/r \end{aligned}\right\} . \qquad (1)$$

We have further to satisfy the identical strain-relation. To obtain this in general conjugate curvilinear coordinates we may proceed as follows, avoiding the consideration of general processes not to be entered on here. Let h be the arcual parameter* of the coordinates ξ, η, and put $\log h = m$, so that $\nabla^2 m = 0$. The strains are

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[*] *Loc. cit.*, p. 106.

[†] *Comptes Rendus*, 1892, p. 1465.

[‡] *Comptes Rendus*, 1892, p. 1510.

[§] Hertz, *Miscellaneous Papers*, XV.

[||] Love, *Elasticity*, I., p. 217.

$$e = \frac{d}{d\xi}\left(\frac{u}{h}\right) + \frac{dm}{d\xi}\,\frac{u}{h} + \frac{dm}{d\eta}\,\frac{v}{h}$$

$$f = \frac{d}{d\eta}\left(\frac{v}{h}\right) + \frac{dm}{d\xi}\,\frac{u}{h} + \frac{dm}{d\eta}\,\frac{v}{h},$$

$$c = \frac{d}{d\xi}\left(\frac{v}{h}\right) + \frac{d}{d\eta}\left(\frac{u}{h}\right).$$

Putting $$\frac{dm}{d\xi}\,\frac{u}{h} + \frac{dm}{d\eta}\,\frac{v}{h} = w,$$

we have $$\frac{dc}{d\xi} - \frac{de}{d\eta} = \frac{d^2}{d\xi^2}\left(\frac{v}{h}\right) - \frac{dw}{d\eta},$$

$$\frac{dc}{d\eta} - \frac{df}{d\xi} = \frac{d^2}{d\eta^2}\left(\frac{u}{h}\right) - \frac{dw}{d\xi}.$$

Hence $$\frac{dc}{d\xi} - \frac{de}{d\eta} + \frac{df}{d\eta} = \nabla_1^2\,\frac{v}{h},$$

$$\frac{dc}{d\eta} - \frac{df}{d\xi} + \frac{de}{d\xi} = \nabla_1^2\,\frac{u}{h},$$

where $$\nabla_1^2 \equiv \frac{d^2}{d\xi^2} + \frac{d^2}{d\eta^2},$$

so that $$\nabla_1^2 m = 0;$$

therefore
$$\begin{aligned}\frac{d^2e}{d\eta^2} + \frac{d^2f}{d\xi^2} - \frac{d^2c}{d\xi\,d\eta} &= \nabla_1^2 w \\ &= \frac{dm}{d\xi}\nabla_1^2\frac{u}{h} + \frac{dm}{d\eta}\nabla_1^2\frac{v}{h}, \\ &\quad + 2\frac{d^2m}{d\xi^2}\,\frac{d}{d\xi}\,\frac{u}{h} + 2\frac{d^2m}{d\eta^2}\,\frac{d}{d\eta}\,\frac{v}{h} \\ &\quad + 2\frac{d^2m}{d\xi\,d\eta}\left(\frac{d}{d\eta}\,\frac{u}{h} + \frac{d}{d\xi}\,\frac{v}{h}\right) \\ &= \frac{dm}{d\xi}\left(\frac{dc}{d\eta} - \frac{df}{d\xi} + \frac{de}{d\xi}\right) \\ &\quad + \frac{dm}{d\eta}\left(\frac{dc}{d\xi} - \frac{de}{d\eta} + \frac{df}{d\eta}\right) \\ &\quad + 2\frac{d^2m}{d\xi^2}(e-f) + 2\frac{d^2m}{d\xi\,d\eta}\,c,\end{aligned}$$

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* The arcual parameter here employed is the reciprocal of that in Love's *Elasticity, and in many other books*. The elements of arc along $d\xi$, $d\eta$ are $h\,d\xi$, $h\,d\eta$.

or
$$\frac{d^2e}{d\eta^2}+\frac{d^2f}{d\xi^2}-\frac{d^2c}{d\xi\,d\eta}=\frac{d}{d\xi}\left\{\frac{dm}{d\xi}(e-f)+\frac{dm}{d\eta}c\right\}$$
$$+\frac{d}{d\eta}\left\{\frac{dm}{d\eta}(f-e)+\frac{dm}{d\xi}c\right\},$$

and this is the relation required.

For plane-polars put

$$\xi=\log r=\varpi,\quad \eta=\theta,\quad m=\log r=\varpi,$$

and the equation becomes

$$\frac{d^2e}{d\theta^2}+\frac{d^2f}{d\varpi^2}-\frac{d^2c}{d\theta\,d\varpi}=\frac{d}{d\varpi}(e-f)+\frac{dc}{d\theta}.$$

In the present problem e, f, c vary as $1/r$ since the stresses do so, and we suppose the linear relations between stress and strain to be independent of r. Hence

$$\frac{de}{d\varpi}=-e,$$

$$\frac{d^2f}{d\varpi^2}=-\frac{df}{d\varpi},$$

$$\frac{d^2c}{d\theta\,d\varpi}=-\frac{dc}{d\theta}.$$

The equation therefore reduces to

$$\frac{d^2e}{d\theta^2}+e=0,$$

so that
$$e=B\cos(\theta+\beta)/r. \tag{2}$$

This equation, together with

$$\left.\begin{aligned}Q&=A\cos(\theta+\alpha)/r\\ U&=A\sin(\theta+\alpha)/r\end{aligned}\right\}, \tag{1}$$

determines the distribution of stress. The relations between the constants are discussed below. The solution holds for any kind of plate provided that the moduluses are not functions of r.

Force applied at an Angle.

2. *If the force is applied at the vertex of a rectilinear angle, we must have $Q=U=0$ over the legs of the angle, supposing them*

free from applied force. The equations (1) then give $Q = U = 0$ throughout the plate. The solution in this case is

$$\left.\begin{aligned} P &= E(\theta)\, e = E(\theta)\, B \cos(\theta+\beta)/r \\ Q &= 0 \\ U &= 0 \end{aligned}\right\}, \qquad (3)$$

where $E(\theta)$ is the Young's modulus in the direction θ. The law of transmission of stress from an angle is, therefore, that the stress is purely radial, is inversely proportional to the distance from the angle, and such that the radial elongation varies as the cosine of its inclination to a fixed line determined by the direction of the force at the angle. This, of course, includes the case of a force applied at a point of a straight boundary discussed, for an isotropic solid, by Flamant[*] and Boussinesq.[*] For the experimental verification of the law in this case, reference may be made to a paper by Carus Wilson, *Phil. Mag.*, Vol. xxxii., p. 481 (1891). It was in connexion with this paper that Flamant and Boussinesq's work was done.

In engineering it is of importance to know the limiting directions of the applied force, within which it produces no radial tension. Confining ourselves now to the isotropic solid, let $\theta = \theta_1$, $\theta = \theta_2$ be the boundaries, so that $\theta_2 - \theta_1 = \gamma$ is the angle of the plate. Take the direction of the applied force F as the initial line. Considering the equilibrium of a sector of the plate, and taking $e = 0$, when $\theta = -\beta$, we have

$$\begin{aligned} F &= EB \int_{\theta_1}^{\theta_2} \sin(\theta+\beta) \cos\theta\, d\theta \\ &= \tfrac{1}{2} EB \left\{ \gamma \sin\beta + \sin\gamma \sin(2\theta_1 + \gamma + \beta) \right\}, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_{\theta_1}^{\theta_2} \sin(\theta+\beta) \sin\theta\, d\theta \\ &= \tfrac{1}{2} \left\{ \gamma \cos\beta - \sin\gamma \cos(2\theta_1 + \gamma + \beta) \right\}. \end{aligned}$$

The latter equation gives

$$\tan\beta = \cot(2\theta_1 + \gamma) - \gamma \operatorname{cosec}\gamma \operatorname{cosec}(2\theta_1 + \gamma),$$

and the former gives B, for the given force. If, now, $P = 0$ when $\theta = \theta_1$, we have $\theta_1 = -\beta$; and therefore

$$\tan\beta = (2\gamma - \sin 2\gamma)/2 \sin^2\gamma.$$

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[*] *Loc. cit.*

When γ is small, $\beta = 2\gamma/3$,
$\gamma = 30^\circ$, $\beta = 19^\circ 55'$,
$\gamma = 60^\circ$, $\beta = 39^\circ 19'$,
$\gamma = 90^\circ$, $\beta = 57^\circ 31'$.

The commonly employed rule of the middle third is therefore accurate for a small angle, and nearly true even for an angle of 90°.

Force applied at an Internal Point.

3. The simple distribution of stress of the last section may be used to build up the solution for other boundaries at which given forces are applied, as we shall see below; but for an internal force we must fall back on the complete solution (1), (2), because the simpler solution will not in general give one-valued displacements in the plate. The conditions that the displacement may be one-valued will determine the constants A, a in the expressions for Q, U, as we proceed to show.

Consider any closed curve s in the plate. Let ϑ be the angle turned through by the tangent at $(x, y)\, s$, and $\sigma\, ds$ the elongation of the element ds at the same point, when the plate is strained. Using, for the moment, rectangular coordinates, the displacement of the point (x_1, y_1) on the curve is given by

$$u_1 = u_0 + \int_0^1 \sigma \cos\psi\, ds - \int_0^1 \vartheta\, dy,$$

$$v_1 = v_0 + \int_0^1 \sigma \sin\psi\, ds + \int_0^1 \vartheta\, dx,$$

or

$$u_1 = u_0 + \int_0^1 \sigma \cos\psi\, ds - \vartheta_0 (y_1 - y_0) + \int_0^1 (y - y_1)\, d\vartheta,$$

$$v_1 = v_0 + \int_0^1 \sigma \sin\psi\, ds + \vartheta_0 (x_1 - x_0) - \int_0^1 (x - x_1)\, d\vartheta,$$

where ψ is the inclination of the tangent to the axis of x.

Let the integration be extended over the whole curve; then, for continuity of displacement we must have

$$\left.\begin{aligned} \int_0^0 \sigma \cos\psi\, ds + \int_0^0 (y - y_0)\, d\vartheta = 0 \\ \int_0^0 \sigma \sin\psi\, ds - \int_0^0 (x - x_0)\, d\vartheta = 0 \end{aligned}\right\}. \qquad (4)$$

Since further
$$\int_0^0 d\vartheta = 0, \tag{5}$$

the conditions (4) reduce to

$$\left.\begin{aligned} \int_0^0 \sigma \cos\psi\, ds + \int_0^0 y\, d\vartheta = 0 \\ \int_0^0 \sigma \sin\psi\, ds - \int_0^0 x\, d\vartheta = 0 \end{aligned}\right\}. \tag{6}$$

We proceed to express (5), (6) in terms of the strains. Let the curve s be one of a system (ξ) of conjugate curvilinears of which the orthogonals are the curves (η). The symbols applying now to these curvilinear coordinates, the strains are given by

$$he = \frac{du}{d\xi} + v\frac{dm}{d\eta},$$

$$hf = \frac{dv}{d\eta} + u\frac{dm}{d\xi},$$

$$hc = \frac{dv}{d\xi} + \frac{du}{d\eta} - u\frac{dm}{d\eta} - v\frac{dm}{d\xi},$$

and the rotation by

$$2h\varpi = \frac{dv}{d\xi} - \frac{du}{d\eta} - u\frac{dm}{d\eta} + v\frac{dm}{d\xi}.$$

Whence, remembering that $\nabla^2 m = 0$, we have

$$2h\frac{d\varpi}{d\eta} = -\frac{dhc}{d\eta} - c\frac{dh}{d\eta} + 2\frac{d}{d\xi}(hf) - 2e\frac{dh}{d\xi}.$$

Now, by a well-known theorem,

$$\vartheta = \varpi - \tfrac{1}{2}c.$$

Hence
$$\frac{d\vartheta}{d\eta} = -\frac{1}{h}\frac{d}{d\eta}(hc) + \frac{1}{h}\frac{d}{d\xi}(hf) - \frac{e}{h}\frac{dh}{d\xi}.$$

The equations (5), (6) therefore become

$$\left.\begin{aligned} &\int_0^0 \left\{\frac{1}{h}\frac{d}{d\xi}(hf) - \frac{1}{h}\frac{d}{d\eta}(hc) - \frac{e}{h}\frac{dh}{d\xi}\right\} d\eta = 0 \\ &\int_0^0 hf\cos\psi\, d\eta + \int_0^0 y\left\{\frac{1}{h}\frac{d}{d\xi}(hf) - \frac{1}{h}\frac{d}{d\eta}(hc) - \frac{e}{h}\frac{dh}{d\xi}\right\} d\eta = 0 \\ &\int_0^0 hf\sin\psi\, d\eta - \int_0^0 x\left\{\frac{1}{h}\frac{d}{d\xi}(hf) - \frac{1}{h}\frac{d}{d\eta}(hc) - \frac{e}{h}\frac{dh}{d\xi}\right\} d\eta = 0 \end{aligned}\right\}.$$

(7)

For the present problem we take (ξ, η) as conjugate polars with the origin at the point of application of the force. Here $\eta = \theta$, $\xi = \log r$, $h = r$, $m = \log r$, $hf = rf$ is independent of r, and $\psi = \frac{\pi}{2} + \theta$. Since

$$\int_0^{2\pi} \frac{d}{d\theta}(hc)\, d\theta = \Big[hc\Big]_0^{2\pi} = 0,$$

the conditions (7) become

$$\int_0^{2\pi} e\, d\theta = 0, \tag{8}$$

$$\int_0^{2\pi} f \sin\theta\, d\theta + \int_0^{2\pi} \sin\theta \left(\frac{dc}{d\theta} + e\right) d\theta = 0,$$

$$\int_0^{2\pi} f \cos\theta\, d\theta - \int_0^{2\pi} \cos\theta \left(\frac{dc}{d\theta} + e\right) d\theta = 0,$$

and the last two may be simplified to

$$\left.\begin{aligned} \int_0^{2\pi} (f \sin\theta - c\cos\theta)\, d\theta + \int_0^{2\pi} e \sin\theta\, d\theta = 0 \\ \int_0^{2\pi} (f \cos\theta + c\sin\theta)\, d\theta + \int_0^{2\pi} e \cos\theta\, d\theta = 0 \end{aligned}\right\}. \tag{9}$$

The condition (8) is satisfied on account of the form (2) of e. The conditions (9) leave two constants arbitrary in the general solution (1), (2), corresponding to the magnitude and direction of the applied force.

For an isotropic plate

$$Ee = P - \sigma Q,$$

$$Ef = Q - \sigma P,$$

$$\mu c = U = A \sin(\theta + \alpha)/r,$$

where σ is now Poisson's ratio. Hence

$$f = Q(1-\sigma^2)/E - \sigma e,$$

or

$$\begin{aligned} 2\mu f &= Q(1-\sigma) - 2\mu\sigma e \\ &= (1-\sigma) A \cos(\theta+\alpha)/r - 2\mu\sigma B \cos(\theta+\beta)/r. \end{aligned}$$

Taking the applied force in the direction $\theta = 0$, we have $\beta = 0$, and *hence, from equations* (9),

$$\alpha = 0,$$

and

$$(1-\sigma)A\int_0^{2\pi}\cos^2\theta\,d\theta+2A\int_0^{2\pi}\sin^2\theta\,d\theta+2\mu(1-\sigma)B\int_0^{2\pi}\cos^2\theta\,d\theta=0,$$

or

$$A=-2\mu\frac{1-\sigma}{3-\sigma}B;$$

therefore

$$P=2\mu\frac{3+\sigma}{3-\sigma}B\cos\theta/r.$$

The solution can therefore be written

$$P=(3+\sigma)\,C\cos\theta/r,$$

$$Q=-(1-\sigma)\,C\cos\theta/r,$$

$$U=-(1-\sigma)\,C\sin\theta/r.$$

If F is the magnitude of the applied force

$$\begin{aligned}F&=-\int_0^{2\pi}P\cos\theta\,r\,d\theta+\int_0^{2\pi}U\sin\theta\,r\,d\theta\\&=-(3+\sigma)\,\pi C-(1-\sigma)\,\pi C\\&=-4\pi C.\end{aligned}$$

Hence

$$\left.\begin{aligned}P&=-(3+\sigma)\,F\cos\theta/4\pi r\\Q&=\quad(1-\sigma)\,F\cos\theta/4\pi r\\U&=\quad(1-\sigma)\,F\sin\theta/4\pi r\end{aligned}\right\}.\qquad(10)$$

The inclination ϕ of a principal axis of stress to the radius r is given by

$$\tan 2\phi=\frac{2U}{P-Q}=-\frac{1-\sigma}{2}\tan\theta,$$

from which the lines of stress can be drawn. For Poisson's isotropic material

$$\tan 2\phi=-\tfrac{3}{8}\tan\theta.$$

For incompressible material

$$\tan 2\phi=-\tfrac{1}{4}\tan\theta,$$

giving the same lines as those for a Poisson-isotropic solid of three dimensions with a force applied at a point.

Circular Plates and Cylinders. (Fig. 1.)

4. For shortness, I will call the distribution of stress

$$\left.\begin{aligned} P &= k\cos\theta/r \\ Q &= 0 \\ U &= 0 \end{aligned}\right\}$$

a *radial distribution* at the origin O, whose *axis* is $\theta = 0$, and whose intensity is k. We may combine any number of distributions of this kind to obtain a solution, provided that the boundary is such that the displacements are one-valued. This condition is, of course, satisfied if the origins are on the boundary of a simply-connected

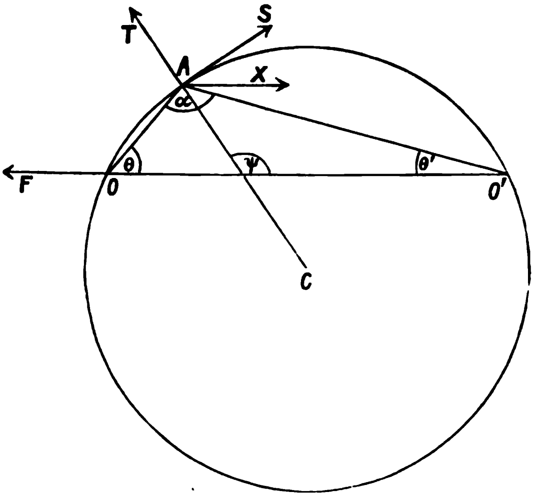

Fig. 1.

plate. Beginning with a single radial distribution, consider the stress across a circle through the origin O. Let the axis of the distribution cut the circle again at O'. Take any point A on the circle and write

$$\begin{aligned} OO' &= c, \\ OA &= r, \\ O'A &= r', \\ \angle OAO' &= \alpha, \\ \angle AOO' &= \theta, \\ \angle AO'O &= \theta'. \end{aligned}$$

If R is the radius of the circle, we have

$$\frac{r}{\sin\theta'} = \frac{r'}{\sin\theta} = \frac{c}{\sin\alpha} = 2R.$$

The stress across the circle at A is in the direction OA, and is equal to

$$k\cos\theta\sin\theta'/r$$

or

$$k\cos\theta/2R,$$

which is equivalent to components

$k\cos\theta\cos\theta'/2R$ along the tangent,

$k\cos\theta\sin\theta'/2R$ along the normal.

Now let ψ be the inclination of the normal to OO', so that

$$\psi = \pi/2+\theta-\theta';$$

then the components become

$k\,(\sin\psi-\cos\alpha)/4R$ along the tangent,

$k\,(\cos\psi+\sin\alpha)/4R$ along the normal.

Now the components

$k\sin\psi/4R$ along the tangent,

$k\cos\psi/4R$ along the normal

are equivalent to $k/4R$ in the direction OO'. Hence, finally, the stress across the circle is composed of three uniform distributions

(*a*) a uniform normal tension $k\sin\alpha/4R$ (T);
(*b*) a uniform tangential shear $-k\cos\alpha/4R$ (S);
(*c*) a uniform tension $k/4R$ (X) along OO', the axis of the distribution.

If OO' is a diameter of the circle, these reduce to
(*a*) a uniform normal tension $k/4R$;
(*b*) a uniform axial tension $k/4R$.

If OO' is a tangent to the circle, the reduction is to
(*a*) a uniform tangential shear $k/4R$;
(*b*) a uniform axial tension $k/4R$.

The signs of these two forces are seen at once by considering the equilibrium of the circle as a whole.

Circular Disc under Two Opposed Forces, or the Roller-Problem. (Fig. 2.)

5. Let a second radial distribution be applied at O' of the same intensity k, but in the opposite direction $O'O$. The stress across the circle is now the sum of the two normal tensions $k \sin \alpha/4R$, the other stresses (b), (c) being equal and opposite for the two distributions. Let this circle be taken as the boundary of the plate, and let accented letters apply to any other circle through OO'. To make the boundary free from stress, except at O, O', we have merely to add to the above distributions a uniform pressure $k \sin \alpha/2R$ throughout the plate. The stress across the circle R' then becomes a tension of magnitude

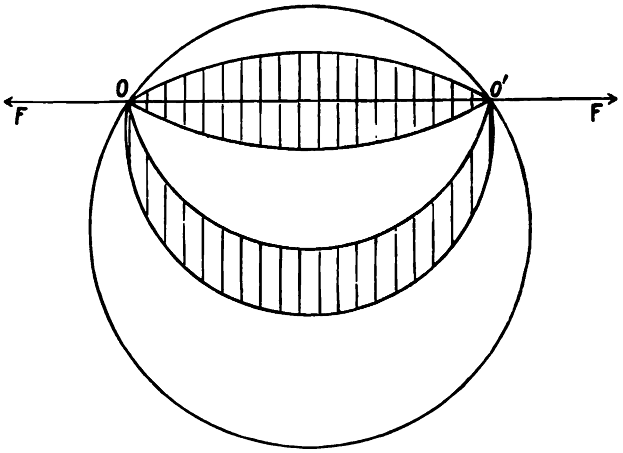

FIG. 2.

$$k\left(\frac{\sin \alpha'}{2R'} - \frac{\sin \alpha}{2R}\right)$$

or

$$\frac{kc}{4}\left(\frac{1}{R'^2} - \frac{1}{R^2}\right).$$

Let F be the thrust applied at O, so that

$$F = -2\pi R \times k/4R = -\pi k/2\,;$$

then we may state the result arrived at as follows:—If a circular plate is acted on by two opposed thrusts F at the ends of a chord OO', *the lines of stress* (tie and strut lines) are the circles on OO'

and the orthogonal circles; the pressure across one of the former circles is constant and equal to

$$\frac{Fc}{2\pi}\left(\frac{1}{R'^2}-\frac{1}{R^2}\right); \tag{11}$$

the pressure across one of the latter circles is

$$\frac{Fc}{2\pi}\left(\frac{\cot\theta\cot\theta'}{R'^2}-\frac{1}{R^2}\right) \tag{12}$$

at the point (r, θ). The last result is obtained from the known sums of principal stresses of the simple distributions combined.

The pressure (11) vanishes when $R = R'$, which happens on the boundary of the plate, and also on its image in the line OO'. This shows that we have here the solution of two other problems of some interest. Taking the lune formed by the arcs α, $\pi-\alpha$ on the same side of OO', we have the solution for the transmission of stress down a semi-crescental plate cantilever loaded at the angle. Taking the arcs α on opposite sides of OO', we have the solution for a spindle-shaped plate strut. The shaded portions of Fig. 2 are examples of the two forms.

If OO' is a diameter of the bounding circle, the expressions (11), (12) both vanish over the boundary. Thus in this case the boundary is entirely free from stress ($P = Q = U = 0$).

Hertz* has considered this last case as an example of his general result for the circular cylinder, but he does not analyse the stress, simply remarking that the magnitudes and directions of the principal stresses can be calculated and that the distribution of stress is extremely complicated.

Circular Disc under a Couple.

(Fig. 3.)

6. Let the equal forces F at O, O' on the circular plate act at right angles to OO' and in opposite directions. The couple formed by these forces is to be balanced by a couple applied at the centre of the disc.

Consider two equal radial distributions at O, O' in the directions opposite to the applied forces F. The uniform normal tensions and

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* *Miscellaneous Papers*, p. 264.

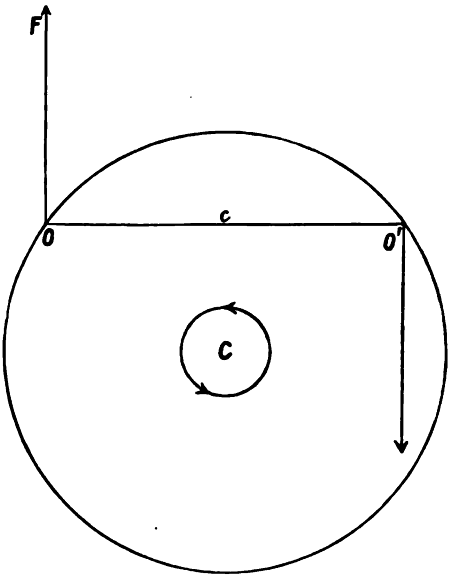

FIG. 3.

the uniform axial tensions over a circle R through OO' disappear, and we are left with a tangential shear $k \sin \alpha / 2R$. Since

$$F = k\pi/2 \quad \text{and} \quad c = 2R \sin \alpha,$$

the uniform shear is $Fc/2\pi R^2$, which of course also follows at once from the consideration that it must balance the couple Fc.

Taking now polar coordinates (ρ, ϑ) at the centre of the disc, we superpose the well known central distribution of stress

$$P = Q = 0,$$

$$U = -Fc/2\pi\rho^2,$$

which transfers the couple balancing the pair of forces F from the circumference of the disc to the centre.

The distribution of stress due to the couple $(F, -F)$ and the opposing couple at the centre is therefore compounded of two radial distributions at O, O' (which give a constant tangential stress $Fc/2\pi R^2$ over each circle through O, O'), and a central system of *pure circumferential* shear of magnitude $Fc/2\pi\rho^2$.

Disc or Solid Wheel loaded at Centre.

7. The circular plate is acted on by two equal and opposite forces F at the ends of a radius CO and along it. Taking polar coordinates (ρ, ϑ) at the centre C with CO as initial line, the distribution of stress in an infinite plate due to F at C is (§ 3)

$$\begin{aligned} P &= -F(3+\sigma)\cos\vartheta/4\pi\rho, \\ Q &= F(1-\sigma)\cos\vartheta/4\pi\rho, \\ U &= F(1-\sigma)\sin\vartheta/4\pi\rho. \end{aligned}$$

With this take the distribution

$$\begin{aligned} P &= A\rho\cos\vartheta, \\ Q &= 3A\rho\cos\vartheta, \\ U &= A\rho\sin\vartheta, \end{aligned}$$

which is immediately seen to verify the resolution-equations with no volume forces

$$\frac{dP}{d\rho}+\frac{1}{\rho}\frac{dU}{d\vartheta}+\frac{P-Q}{\rho}=0,$$

$$\frac{dU}{d\rho}+\frac{1}{\rho}\frac{dQ}{d\vartheta}+\frac{2U}{\rho}=0,$$

and also satisfies $$\nabla^2(P+Q)=0,$$

so that it is a distribution due to boundary forces only. Choose A so that the combined stresses give a uniform distribution in the direction CO over the boundary $\rho = R$. This requires

$$AR-F(3+\sigma)/4\pi R = -AR-F(1-\sigma)/4\pi R$$

or $$A = F(1+\sigma)/4\pi R^2.$$

We then have at $\rho = R$

$$\begin{aligned} P &= -F\cos\vartheta/2\pi R, \\ Q &= F(2+\sigma)\cos\vartheta/2\pi R, \\ U &= F\sin\vartheta/2\pi R, \end{aligned}$$

and the stress across this circle is a uniform distribution $F/2\pi R$ in the direction OC. Now the radial distribution at O, corresponding to the force F at that point, gives a uniform distribution over the circle the opposite of the former, together with a normal pressure $F/2\pi R$, which is got rid of as before by the addition of a uniform

$$\text{[illegible]} = \frac{\text{[illegible]}}{4\pi}\left\{\frac{\text{[illegible]}}{\rho} + (1+\sigma)\frac{\rho}{R^2}\right\}\sin\vartheta.$$

e radial distribution at O.

$$P = -\frac{2F}{\pi}\,\frac{\cos\theta}{r},$$

$$Q = 0,$$

$$U = 0.$$

A Crank Disc.

n a very similar manner we can treat the p
disc in so far as it is a problem of plane stress.
thrust, already considered, we have a tangenti
nt O on the circumference, an equal and opposi
C, and a couple FR at the centre. Taking the
ection of the force F at the centre, the last
e system

$$P = -\frac{F}{4\pi}\left\{\frac{3+\sigma}{\rho} - (1+\sigma)\frac{\rho}{R^2}\right\}\cos\vartheta,$$

$$Q = \frac{F}{4\pi}\left\{\frac{1-\sigma}{\rho} + 3(1+\sigma)\frac{\rho}{R^2}\right\}\cos\vartheta,$$

$$U = \frac{F}{4\pi}\left\{\frac{1-\sigma}{\rho} + (1+\sigma)\frac{\rho}{R^2}\right\}\sin\vartheta + \frac{F}{\text{[illegible]}}$$

Heavy Disc or Roller.

9. To find the stress in a heavy disc or roller on a horizontal plane, consider the following elementary solution expressed *in rectangular coordinates* with origin at the centre C, and axis of x vertically upwards,

$$P' = \tfrac{1}{2}wx,$$
$$Q' = -\tfrac{1}{2}wx,$$
$$U' = \tfrac{1}{2}wy.$$

These give
$$P'_x + U'_y = w,$$
$$U'_x + Q'_y = 0,$$
and
$$\nabla^2 (P' + Q') = 0.$$

The distribution is therefore due to a uniform vertical (downward) force w per unit area, and a uniform vertical (upward) force $\tfrac{1}{2}wR$ over the boundary. Now the radial distribution at O, the point of contact of the disc with the plane, corresponding to the pressure πwR^2, gives a uniform downward pressure $\tfrac{1}{2}wR$ over the boundary, together with a uniform normal pressure, annulled as before. Hence the stress in a heavy disc or roller is compounded of the rectangular system (origin at centre)

$$P' = \tfrac{1}{2}w(x+R),$$
$$Q' = -\tfrac{1}{2}w(x-R),$$
$$U' = \tfrac{1}{2}wy,$$

and the radial distribution at O

$$P = -2wR^2 \cos\theta / r,$$
$$Q = 0,$$
$$U = 0.$$

If we take a horizontal section of the disc, the stress across it is directed radially from O, and its magnitude is

$$-\frac{w}{2r}(2R\cos\theta - r)(2R\cos\theta + r).$$

If we take a radial section through O, the stress across it is a horizontal tension of magnitude

$$\tfrac{1}{2}w(2R\cos\theta - r).$$

The boundary is entirely free from stress ($P = Q = U = 0$) except at O.

General Problem of Circular Plate or Cylinder.

10. Consider any number of simple radial distributions with their origins on a circle of radius R whose centre is C. Each one gives (§ 4) over the circle a uniform normal tension $F\sin\alpha/2\pi R$, a uniform tangential shear $-F\cos\alpha/2\pi R$, and a uniform tension $F/2\pi R$ in the direction of the radial distribution whose intensity is $2F/\pi$, corresponding to a force F on the circle in the opposite direction at the origin. Now $F\sin\alpha$, $F\cos\alpha$ are the components of F along radius and tangent respectively, at its point of application. If we take moments about C, we see that $\Sigma F\cos\alpha = 0$, if the forces F form a balancing system. The uniform tensions $F/2\pi R$ in the directions of the forces F give a zero resultant at each point of the boundary for a like reason. We are left with a sum of normal tensions $\Sigma F\sin\alpha/2\pi R$ at each point of the boundary. This we may write $T/2\pi R$, where T is the sum of the normal (outward) components of the applied forces. This uniform tension we get rid of as before, and so obtain the following theorem :—

If a circular plate is acted on by any balancing system of forces ΣF applied to its boundary, the stress at any point is that due to the radial distributions, such as $2F\cos\theta/\pi r$ at the points of application of the forces ΣF, and in the same directions as those forces, together with a uniform pressure of magnitude $T/2\pi R$, where T is the sum of the normal tensions on the boundary.

This theorem is due to Hertz,[*] who gave an analytical proof of it.

Circular Hole in an Infinite Plate.

11. A similar process can be applied to the problem of any applied forces on a circular hole in an infinite plate, provided these forces either balance or form a couple. The limitation is not serious, as any simple solution giving a resultant force over the hole removes it. For example, the solution of § 3 is sufficient for the purpose.

It is readily shown that the system of radial distributions corresponding to such a system of applied forces gives one-valued displacements in a region not enclosing any of the points of application.

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[*] *Loc. cit.*

Take one of the distributions $2F\cos(\theta+\alpha)/\pi r$, and consider the integrals*

$$\int_0^0 \frac{dP}{dn}\,ds,$$

$$\int_0^0 \left(x\frac{dP}{ds}-y\frac{dP}{dn}\right)ds,$$

$$\int_0^0 \left(y\frac{dP}{ds}+x\frac{dP}{dn}\right)ds,$$

taken over the circle r.

The first integral vanishes, the second and third are respectively $-4F\sin\alpha$, $-4F\cos\alpha$, and these will be the values over any other closed curve surrounding the origin. Hence, for the whole system of radial distributions, inside the closed curve integrated over, the above integrals vanish, for by assumption $\Sigma F\sin\alpha$, $\Sigma F\cos\alpha$ vanish. The conditions for a one-valued displacement are therefore satisfied. Let S be the algebraic sum of the tangential forces over the hole. The distribution of stress in the plate is that due to the radial distributions $2F\cos\theta/\pi r$, together with a pure circumferential shear of magnitude $SR/2\pi\rho^2$, a radial tension of magnitude $TR/2\pi\rho^2$, and an equal circumferential pressure. The last distribution, introduced to annul a uniform pressure over the hole, is equivalent to a pure shear on planes making an angle of 45° with the radius, and therefore gives shear lines in the form of equiangular spirals. It seems probable that the special curves sometimes produced around a punched hole in a plate are due to such a radial pressure produced by the lateral spread of the material under the punch. See, for example, the frontispiece in Todhunter and Pearson's *History of Elasticity*, Vol. II., Part 2. This explanation would correspond with the theory which Tresca has given for the flow of metal around a punch-hole.†

APPENDIX.‡

Some Diagrams of Plane Stress.

The accompanying diagrams are intended to illustrate some of the solutions given in my paper on "Elementary Distributions of Plane Stress," recently communicated to the Society. The stress is defined by a single function ψ, the stress-function; but the construction of the curves $\psi = \text{const.}$ would not give a convenient graphical representation of the stress, since second derivatives of ψ are involved in its expression. The curves $\psi_x = \text{const.}$, $\psi_y = \text{const.}$, which are drawn in the present diagrams, indicate the stress at each point simply and completely.

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* Previous paper, Vol. XXXI., p. 108. P is here the $\nabla^2\psi\ (=P+Q)$ of that paper, since $Q=0$.

† Flamant, *Résistance des Matériaux*, p. 296.

‡ [*This Appendix to Mr. Michell's paper* was received in May, 1900; it is *printed here for convenience of reference.—Sec.*]

The stress* across the curve $\psi_x = \text{const.}$ is everywhere parallel to the axis of x, and its intensity at any point is the rate of increase of ψ_y along the curve. The stress across a line parallel to the axis of x is, of course, conjugate to that across $\psi_x = \text{const.}$, that is, it is in the direction of the tangent to that curve. Its intensity at any point is the normal rate of increase of ψ_x or is measured by the closeness of the consecutive curves $\psi_x = \text{const.}$ The stress at (x, y) is completely defined by these conjugate stresses. It is similarly defined by the conjugate stresses across $\psi_y = \text{const.}$ and $x = \text{const.}$ The rectangular elements of stress are represented in the figures by the rates of increase of ψ_x, ψ_y in the directions of the axes, viz.,

$$P = \frac{d}{dy}(\psi_y),$$

$$Q = \frac{d}{dx}(\psi_x),$$

$$U = -\frac{d}{dx}(\psi_y) = -\frac{d}{dy}(\psi_x).$$

These equations are also conveniently used to determine the signs of the conjugate stresses considered.

The curves were drawn by superposition of those for the elementary distributions employed in the solutions of the various cases. It will be well to give a list of the component and the final curves.

Components.

1. *Simple Radial Distribution.*—Taking polar coordinates at the point of application of the force F and the initial line (axis of x) in the direction of that force,

$$\psi^1 = -\frac{F}{\pi} r\theta \sin\theta = -\frac{F}{\pi}\theta y,$$

$$\psi_x^1 = \frac{F}{2\pi}(1-\cos 2\theta),$$

$$\psi_y^1 = -\frac{F}{2\pi}(2\theta+\sin 2\theta).$$

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* *See previous paper*, *Proceedings*, Vol. xxxi., p. 110.

2. *Force at a Point of an Infinite Plate.*—

$$\psi^{\text{ii}} = -\frac{F}{2\pi} r\theta \sin\theta + \frac{F}{4\pi}(1-\sigma)\, r\log r\cos\theta,$$

$$\psi^{\text{ii}}_x = \frac{F}{4\pi}(1-\cos 2\theta) + \frac{F}{8\pi}(1-\sigma)(2\log r + 1 + \cos 2\theta),$$

$$\psi^{\text{ii}}_y = -\frac{F'}{4\pi}(2\theta + \sin 2\theta) + \frac{F}{8\pi}(1-\sigma)\sin 2\theta.$$

3. *Complementary Linear Stress.*—

$$\psi^{\text{iii}} = \frac{F}{8\pi R^2}(1+\sigma)\, r^3\cos\theta,$$

$$\psi^{\text{iii}}_x = \frac{F}{8\pi R^2}(1+\sigma)(3x^2+y^2),$$

$$\psi^{\text{iii}}_y = \frac{F}{4\pi R^2}(1+\sigma)\, xy.$$

4. *Uniform Tension T.*—

$$\psi^{\text{iv}} = \tfrac{1}{2}Tr^2,$$

$$\psi^{\text{iv}}_x = Tx,$$

$$\psi^{\text{iv}}_y = Ty.$$

5. *Symmetrical Shear due to Couple L at Origin.*—

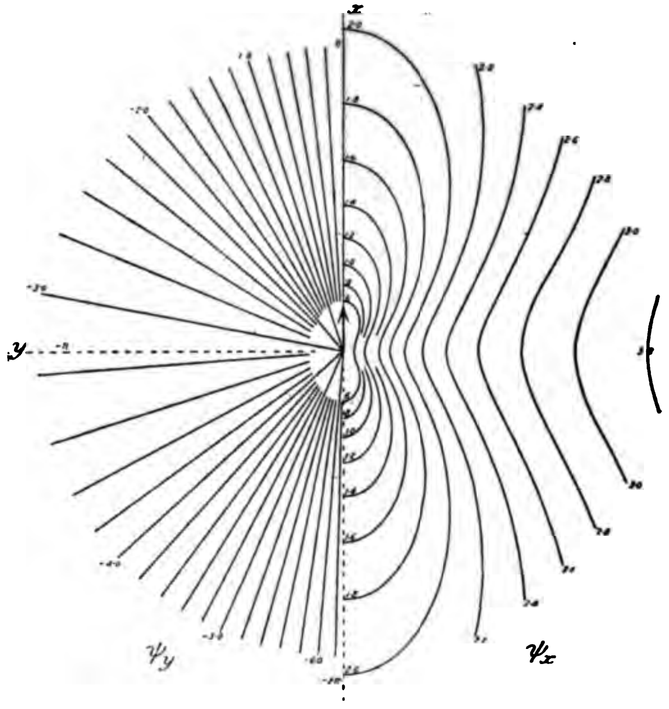
$$\psi^{\text{v}} = \frac{L}{2\pi}\theta,$$

$$\psi^{\text{v}}_x = -\frac{L}{2\pi}\frac{\sin\theta}{r},$$

$$\psi^{\text{v}}_y = \frac{L}{2\pi}\frac{\cos\theta}{r}.$$

The Diagrams.

The axis of x is always taken in the direction of one of the applied forces which are indicated in the diagrams by arrowheads. Polar coordinates having their origin at the centre of the disc are written (ρ, ϑ), those having their origin at the points of application of the forces on the boundary (r, θ), suffixes being used where necessary. The initial line at each origin is always in the direction of the axis of x as indicated in each diagram. On account of the symmetry with respect to a diameter of the disc, it is sufficient to represent the curves $\psi_x =$ const. on one half, and the curves $\psi_y =$ const. on the other. This has been done except in the diagrams V. (*a*), (*b*). The full lines represent curves proceeding by equal differences; intermediate curves are drawn as broken lines and their parameters indicated. The scale on which the stresses are indicated by the numbers in the diagrams is derived by considering a complete section of the disc. Thus, in Diagram II., the resultant stress across a horizontal section must be equal to one of the applied forces which is therefore represented by 2π.

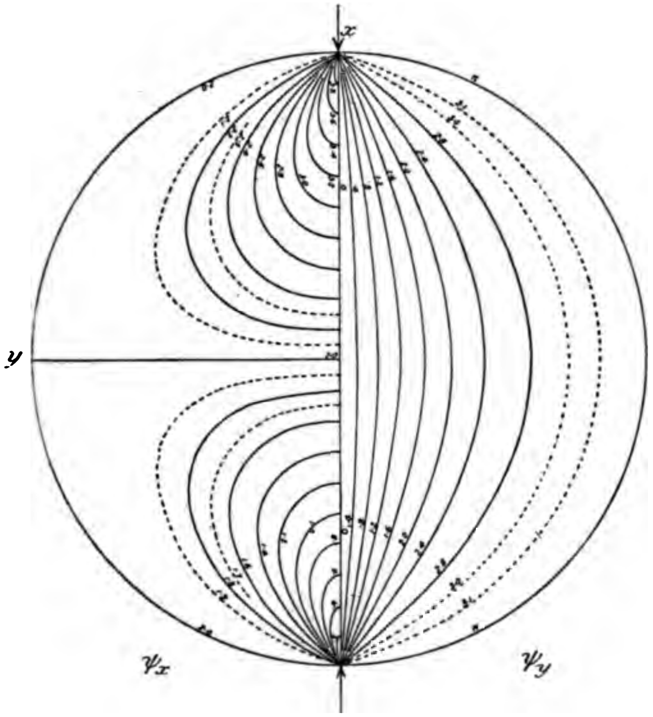


I. *Force applied at a Point of an Infinite Plate* ($\sigma = \frac{1}{4}$).

The curves are

$$\psi_x = \frac{F}{4\pi}(1-\cos 2\theta) + \frac{3F}{32\pi}(2\log r + 1 + \cos 2\theta),$$

$$\psi_y = -\frac{F}{4\pi}(2\theta + \sin 2\theta) + \frac{3F}{32\pi}\sin 2\theta.$$



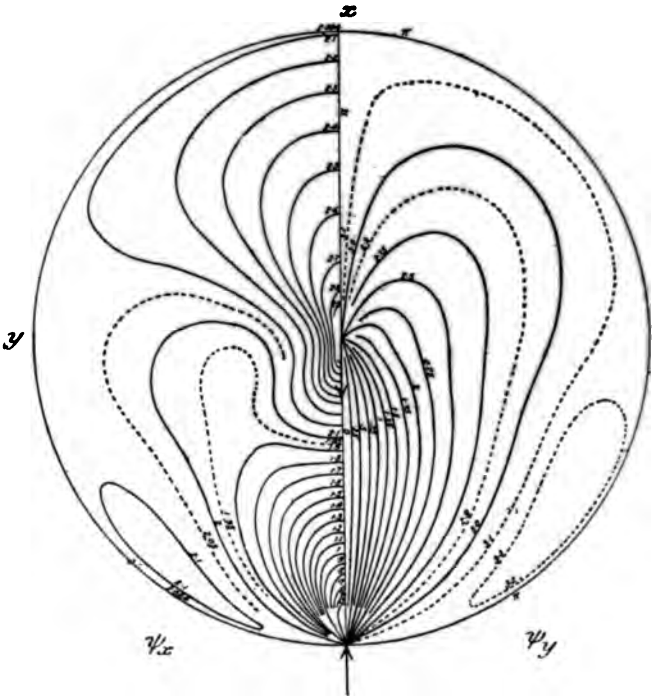
II. *Roller.*

The applied forces are at opposite ends of a diameter.

$$\psi_x = \frac{F}{2\pi}(1-\cos 2\theta_2) - \frac{F}{2\pi}(1-\cos 2\theta_1) + \frac{F}{\pi R}\rho \cos \vartheta,$$

$$\psi_y = -\frac{F}{2\pi}(2\theta_2+\sin 2\theta_2) + \frac{F}{2\pi}(2\theta_1+\sin 2\theta_1) + \frac{F}{\pi R}\rho \sin \vartheta,$$

where R is the radius of the disc, and the suffix 1 refers to the point ($y = 0$, $x = R$).

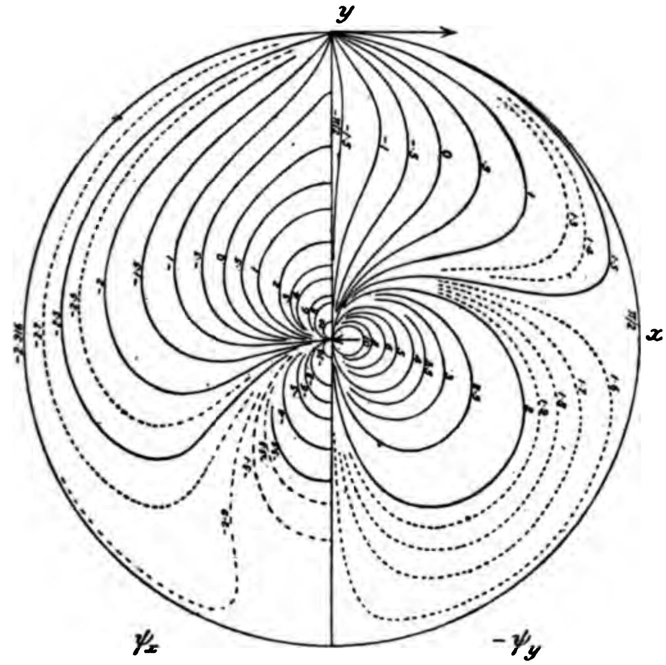


III. *Disc Wheel.*

Here a force F at the centre of the disc is balanced by an opposite force at the circumference.

$$\psi_r = \frac{F}{2\pi}(1-\cos 2\theta) - \frac{F}{4\pi}(1-\cos 2\vartheta) - \frac{3F}{32\pi}(2\log\rho + 1 + \cos 2\vartheta)$$
$$- \frac{5F}{32\pi R^2}\rho^2(2+\cos 2\vartheta) + \frac{F}{2\pi R}\rho\cos\vartheta,$$

$$\psi_\nu = -\frac{F}{2\pi}(2\theta+\sin 2\theta) + \frac{F}{4\pi}(2\vartheta+\sin 2\vartheta) - \frac{3F}{32\pi}\sin 2\vartheta$$
$$- \frac{5F}{32\pi R^2}\rho^2\sin 2\vartheta + \frac{F}{2\pi R}\rho\sin\vartheta.$$

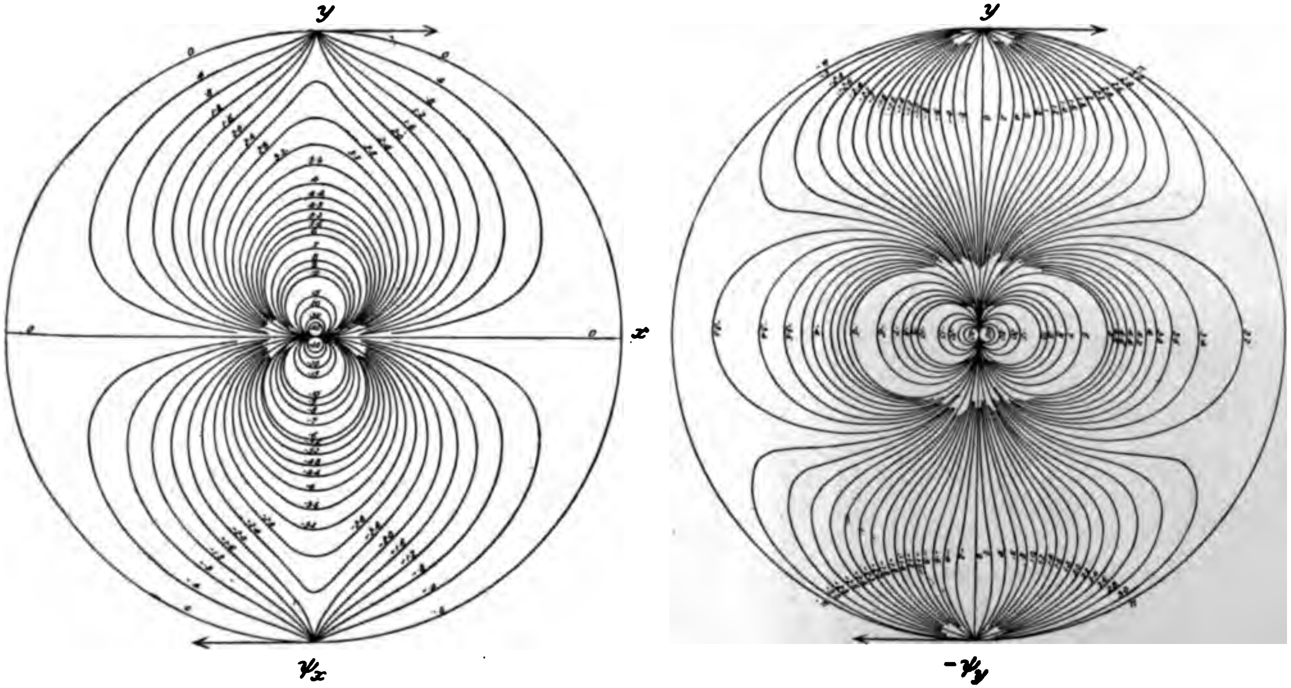


IV. *Disc-Crank.*

A tangential force F at the point ($x = 0$, $y = R$) on the circumference is balanced by a force and a couple at the centre.

$$\psi_x = \frac{F}{2\pi}(1-\cos 2\theta) - \frac{F}{4\pi}(1-\cos 2\vartheta) - \frac{3F}{32\pi}(2\log\rho + 1 + \cos 2\vartheta)$$
$$- \frac{5F}{32\pi R^2}\rho^2(2+\cos 2\vartheta) + \frac{FR}{2\pi}\,\frac{\sin\vartheta}{\rho},$$

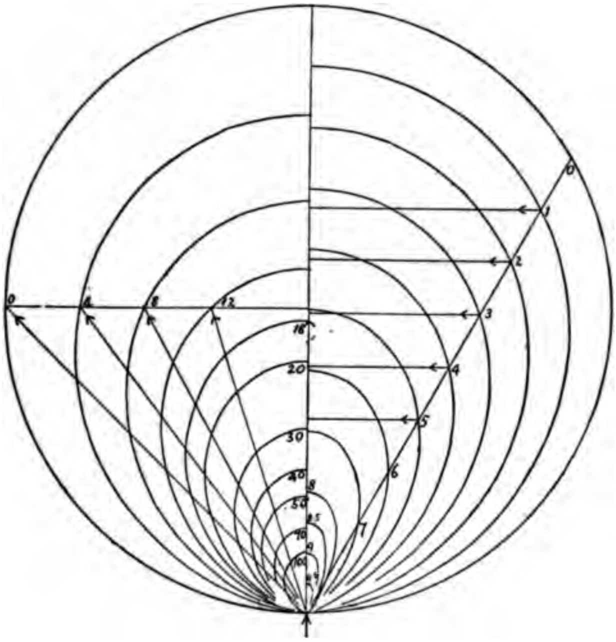
$$\psi_y = -\frac{F}{2\pi}(2\theta + \sin 2\theta) + \frac{F}{4\pi}(2\vartheta + \sin 2\vartheta) - \frac{3F}{32\pi}\sin 2\vartheta$$
$$- \frac{5F}{32\pi R^2}\rho^2 \sin 2\vartheta - \frac{FR}{2\pi}\,\frac{\cos\vartheta}{\rho}.$$



V. (*a* and *b*). *Wrist-Plates.*

Here two tangential forces F at ($x = 0$, $y = \pm R$) are balanced by a couple at the centre.

$$\psi_x = \frac{F}{2\pi}(1-\cos 2\theta_1) - \frac{F}{2\pi}(1-\cos 2\theta_2) + \frac{FR}{\pi}\,\frac{\sin\vartheta}{\rho}, \qquad \psi_y = -\frac{F}{2\pi}(2\theta_1 + \sin 2\theta_1) + \frac{F}{2\pi}(2\theta_2 + \sin 2\theta_2) - \frac{FR}{\pi}\,\frac{\cos\vartheta}{\rho}.$$



VI. *Heavy Disc.*

The disc is in a vertical plane and is supported at its lowest point. The stress-function in this case is not very convenient for the representation of the stress, on account of the volume-forces. The figure represents a set of conjugate stresses. The stress across any line through the point of support is horizontal, and its intensity at any point is given by the parameter of the curve on the right which passes through the point. The stress across any horizontal line is directed radially from the point of support, and its intensity at any point is given by the parameter of the curve on the left, which passes through the point.

A Formula in the Theory of Single Theta-Functions. **By A. C. Dixon.**

Communicated February 8th, 1900. Received, in revised form, April 16th, 1900.

The addition-theorem for three arguments in the case of the functions sn u, cn u, dn u is perhaps most compactly given by the formula

$$\begin{aligned} &Hu_1 Hu_2 Hu_3 Hu_4 - \Theta u_1 \Theta u_2 \Theta u_3 \Theta u_4 \\ &\qquad - H(u_1+K)\,H(u_2+K)\,H(u_3+K)\,H(u_4+K) \\ &\qquad + \Theta(u_1+K)\,\Theta(u_2+K)\,\Theta(u_3+K)\,\Theta(u_4+K) = 0, \end{aligned}$$

where u_1, u_2, u_3, u_4 are any four arguments whose sum is zero. For, by such substitutions as

$$u_1+K,\ u_2-K,\ u_3,\ u_4 \quad \text{for} \quad u_1,\ u_2,\ u_3,\ u_4,$$

other formulæ may be derived from this one, and thus the sn, cn, dn of u_4 expressed in terms of those of u_1, u_2, u_3 in many ways.

For the functions

$$Hu : H(u+\tfrac{4}{3}K) : H(u+\tfrac{8}{3}K)$$

there is a corresponding theorem, namely, that, if $u_1+u_2+u_3=0$, then

$$\begin{aligned} &Hu_1 Hu_2 Hu_3 + H(u_1+\tfrac{4}{3}K)\,H(u_2+\tfrac{4}{3}K)\,H(u_3+\tfrac{4}{3}K) \\ &\qquad + H(u_1+\tfrac{8}{3}K)\,H(u_2+\tfrac{8}{3}K)\,H(u_3+\tfrac{8}{3}K) = 0. \end{aligned}$$

[*Quarterly Journal*, Vol. XXIV., p. 181 (37*a*), p. 225 (76).]

Again, if u_1, u_2, ..., u_n are n angles whose sum is $\frac{\pi}{2}$,

$$\Pi \cos u - \Pi \cos\left(u+\frac{\pi}{n}\right) + \Pi \cos\left(u+\frac{2\pi}{n}\right) \ldots \text{ to } n \text{ terms} = 0.$$

These are instances of a general theorem which it is the object of this note to investigate, that, if θu is a holomorphic function of u, such that

$$\theta(u+2\omega) = e^{\alpha u+\beta}\,\theta u,$$

$$\theta(u+2\omega') = e^{\alpha' u+\beta'}\,\theta u,$$

and having only a simple zero within the parallelogram of the periods 2ω, $2\omega'$, namely, when $u = 0$, and if $\theta_{rs}u$ denotes

$$\exp\left\{-\left(\frac{ra}{m}+\frac{sa'}{n}\right)\left(u+\frac{r\omega}{m}+\frac{s\omega'}{n}\right)+\frac{r}{m}(a\omega-\beta)+\frac{s}{n}(a'\omega'-\beta')\right\}$$
$$\times\,\theta\left(u+\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right)$$

where r, s, m, n are any integers, m, n being positive, then

$$\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(-1)^{rs+rn(m-1)+sm(n-1)}\prod_{i=1}^{mn}\theta_{rs}u_i = 0$$

if $u_1, u_2, \ldots, u_{mn}$ are mn arguments whose sum

$$\equiv (mn-1)(n\omega+m\omega') \quad (\text{mod } 2n\omega,\ 2m\omega').$$

This formula contains implicitly the addition-theorem for $mn-1$ arguments relating to the functions of the type

$$\theta_{rs}u/\theta u$$

which are so important in the theory of transformation.

When $m = n = 2$ it is possible to express

$$\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(-1)^{rs+rn(m-1)+sm(n-1)}\prod_{i=1}^{mn}\theta_{rs}u_i$$

as the product of factors each a theta-function. This does not appear to be possible in general. It can be done when $n = 1$, $m = 2$, and for a particular value of the modulus when $n = 1$, $m = 3$, and, again, for the trigonometrical case given above for any value of n. The form of the results, especially when $mn = 3$, seems to show that they are exceptional.

Proof of the Formula.

Taking periods 2ω, $2\omega'$, let θu be a theta-function, having only a simple zero within the period parallelogram, namely, for the value 0 of the argument, and such that

$$\theta(u+2\omega) = e^{au+\beta}\theta u,$$
$$\theta(u+2\omega') = e^{a'u+\beta'}\theta u.$$

Then it is known that $a\omega'-a'\omega = \iota\pi$, ι being the sign of the imaginary part of ω'/ω.

Consider the function

$$\phi_{rs}u \equiv e^{-(ra/m+sa'/n)u}\,\frac{\theta\left(u+\dfrac{2r\omega}{m}+\dfrac{2s\omega'}{n}\right)}{\theta\left(\dfrac{2r\omega}{m}+\dfrac{2s\omega'}{n}\right)},$$

where m, n, r, s are any four integers, the first two of which will be supposed positive and kept fixed throughout the discussion; in the exceptional case when $\frac{r}{m}$, $\frac{s}{n}$ are both integral, we shall suppose the denominator of $\phi_{rs}u$ to be $\theta'\left(\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right)$; similarly in like cases throughout. It is then easily found that

$$\phi_{r+m,\,s}\,u = \phi_{rs}u,$$

$$\phi_{r,\,s+n}\,u = \phi_{rs}u,$$

so that there are only mn functions such as $\phi_{rs}u$.

Also

$$\frac{\phi_{rs}(u+2\omega)}{\theta(u+2\omega)} \div \frac{\phi_{rs}u}{\theta u}$$

$$= \exp\left\{-\left(\frac{r\alpha}{m}+\frac{s\alpha'}{n}\right)2\omega+\alpha\left(u+\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right)+\beta-(\alpha u+\beta)\right\}$$

$$= \exp\frac{2s\iota\pi}{n};$$

and, in like manner,

$$\frac{\phi_{rs}(u+2\omega')}{\theta(u+2\omega')} \div \frac{\phi_{rs}u}{\theta u} = \exp\left(-\frac{2r\iota\pi}{m}\right).$$

Thus the function $\frac{\phi_{rs}u}{\theta u}$ has the two periods $2n\omega$ and $2m\omega'$, and within its parallelogram the mn poles

$$2i\omega+2j\omega' \quad (i=0,\,1,\,\ldots,\,n-1;\ j=0,\,1,\,\ldots,\,m-1);$$

the same applies to the more general function

$$\sum_{r=0}^{m-1}\sum_{s=0}^{n-1} A_{rs}\phi_{rs}u \div \theta u,$$

in which the coefficients A are any constants. This function has therefore the value zero for mn values of u, whose sum is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(2i\omega+2j\omega'),$$

that is, $$mn(n-1)\,\omega+nm(m-1)\,\omega',$$

this being the sum of the values of u at its poles. Now the mn

coefficients A may be so determined that $\Sigma\Sigma A_{rs}\phi_{rs}u$ shall vanish for $mn-1$ distinct given arguments

$$u_1,\ u_2,\ \ldots,\ u_{mn-1}.$$

It will then vanish for only one more, namely, u_{mn}, where

$$u_1+u_2+\ldots+u_{mn} = mn(n-1)\omega+nm(m-1)\omega',$$

so that, if there are mn arguments

$$u_1,\ u_2,\ \ldots,\ u_{mn},$$

whose sum $\equiv mn(n-1)\omega+nm(m-1)\omega' \pmod{2n\omega,\ 2m\omega'}$,

the determinant of the m^2n^2 quantities

$$\phi_{rs}u_k$$

must vanish; and, conversely, if this determinant vanishes, the sum of the arguments is as stated. We may write this determinant as

$$\Sigma\left(\pm\prod_{h=1}^{mn}\phi_{r_h s_h}u_h\right).$$

Now, we may put $u_i+2k\omega$, $u_j-2k\omega$ for u_i, u_j without affecting the sum of the arguments; the determinant thus becomes

$$e^{ka(u_i-u_j+2k\omega)}\,\Sigma\left\{\pm e^{(s_i-s_j)2ki\pi/n}\prod_{h=1}^{mn}\phi_{r_h s_h}u_h\right\},$$

which must vanish for the values 0, 1, 2, ..., $n-1$ of k. From the n equations thus derived it follows that the sum of those terms in the determinant for which $s_i-s_j \pmod{n}$ has any particular value must vanish separately. Any one of the parts into which the whole expression is thus divided may now be treated similarly by picking out any two arguments, increasing the one by a multiple of 2ω or $2\omega'$, and diminishing the other by an equal quantity. In this way the determinant is separated into parts, each of which must vanish separately; in each part all the differences such as s_i-s_j, r_i-r_j are the same (mod n, m respectively) in each term, and, using a double suffix notation for the arguments, we have the following result:—

$$\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\left(\pm\prod_{i=0}^{m-1}\prod_{j=0}^{n-1}\phi_{i+r,\,j+s}u_{i,j}\right)=0.$$

It is now possible to give the proper sign to each term of the expression; an addition of 1 to r causes cyclic interchanges in n groups of m factors each, and thus changes the sign $n(m-1)$ times; by symmetry, an addition of 1 to s changes the sign $m(n-1)$ times. Thus the result is

$$\sum_r \sum_s (-1)^{rn(m-1)+sm(n-1)} \prod_{i,j} \phi_{i+r,j+s}\, u_{i,j} = 0.$$

In this equation put

$$u_{ij} = v_{ij} - \frac{2i}{m}\omega - \frac{2j}{n}\omega',$$

so that

$$\sum_{i,j} v_{ij} \equiv \sum_{i,j} u_{ij} + n(m-1)\omega + m(n-1)\omega'$$

$$\equiv n(mn-1)\omega + m(mn-1)\omega'.$$

Now

$$\frac{\phi_{i+r,j+s}\left(v - \dfrac{2i}{m}\omega - \dfrac{2j\omega'}{n}\right)}{\phi_{rs} v}$$

$$= \exp\left\{\left(\frac{ra}{m} + \frac{sa'}{n}\right)v - \left(\frac{i+r}{m}a + \frac{j+s}{n}a'\right)\left(v - \frac{2i}{m}\omega - \frac{2j\omega'}{n}\right)\right\}$$

$$\times\theta\left(\frac{2r\omega}{m} + \frac{2s\omega'}{n}\right) \div \theta\left(2\frac{i+r}{m}\omega + 2\frac{j+s}{n}\omega'\right).$$

Thus

$$\prod e^{(ia/m+ja'/n)[v_{ij}-(2i\omega/m)-(2j\omega'/n)]}\, \phi_{i+r,j+s}\left(v_{ij} - \frac{2i}{m}\omega - \frac{2j}{n}\omega'\right)$$

$$= \prod_{i,j} \phi_{rs} v_{ij}$$

$$\times \frac{\left\{\theta\left(\dfrac{2r\omega}{m} + \dfrac{2s\omega'}{n}\right)\right\}^{mn}}{\prod\limits_{i,j}\theta\left(2\dfrac{i+r}{m}\omega + 2\dfrac{j+s}{n}\omega'\right)} \exp\left[\left(\frac{ra}{m} + \frac{sa'}{n}\right)\{n(m-1)\omega + m(n-1)\omega'\}\right].$$

To compare the denominators in expressions of this type, we have

$$\prod_{i,j}\theta\left(2\frac{i+r}{m}\omega + 2\frac{j+s}{n}\omega'\right) \div \prod_{i,j}\theta\left(2\frac{i}{m}\omega + 2\frac{j+s}{n}\omega'\right)$$

$$= \prod_{i=0}^{r-1}\prod_{j=0}^{n-1} \frac{\theta\left(2\dfrac{i+m}{m}\omega + 2\dfrac{j+s}{n}\omega'\right)}{\theta\left(2\dfrac{i}{m}\omega + 2\dfrac{j+s}{n}\omega'\right)}$$

$$= \exp\sum_{i=0}^{r-1}\sum_{j=0}^{n-1}\left(\frac{2i}{m}a\omega + 2\frac{j+s}{n}a\omega' + \beta\right)$$

$$= \exp\left\{\frac{n}{m}r(r-1)a\omega + r(n-1+2s)a\omega' + nr\beta\right\}$$

$$\prod_{i,j}\theta\left(2\frac{i}{m}\omega+2\frac{j+s}{n}\omega'\right)\div\prod_{i,j}\theta\left(2\frac{i}{m}\omega+2\frac{j}{n}\omega'\right)$$

$$=\prod_{i=0}^{m-1}\prod_{j=0}^{s-1}\frac{\theta\left(2\frac{i}{m}\omega+2\frac{j+n}{n}\omega'\right)}{\theta\left(\frac{2i}{m}\omega+\frac{2j}{n}\omega'\right)}$$

$$=\exp\sum_{i=0}^{m-1}\sum_{j=0}^{s-1}\left(\frac{2i}{m}a'\omega+\frac{2i}{n}a'\omega'+\beta'\right)$$

$$=\exp\left\{s(m-1)a'\omega+\frac{m}{n}s(s-1)a'\omega'+ms\beta'\right\}.$$

Hence

$$\prod_{i,j}e^{(ia/m+ja'/n)[v_{ij}-(2i\omega/m)-(2j\omega')/n]}\phi_{i+r,j+s}\left(v_{ij}-\frac{2i}{m}\omega-\frac{2j}{n}\omega'\right)$$

$$\times\prod_{i,j}\theta\left(\frac{2i}{m}\omega+\frac{2j}{n}\omega'\right)\div\left\{\theta\left(\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right)\right\}^{mn}\prod_{i,j}\phi_{rs}v_{ij}$$

$$=\exp\left[\left(\frac{ra}{m}+\frac{sa'}{n}\right)\left\{n(m-1)\omega+m(n-1)\omega'\right\}-\frac{n}{m}(r^2-r)a\omega\right.$$

$$\left.-r(n-1+2s)a\omega'-nr\beta-s(m-1)a'\omega-\frac{m}{n}s(s-1)a'\omega'-ms\beta'\right]$$

$$=\exp\left\{rs(a'\omega-a\omega')-mn\left(\frac{ra}{m}+\frac{sa'}{n}\right)\left(\frac{r\omega}{m}+\frac{s\omega'}{n}\right)\right.$$

$$\left.+nr(a\omega-\beta)+ms(a'\omega'-\beta')\right\}.$$

Thus
$$\prod_{i,j}\phi_{i+r,j+s}\left(v_{ij}-\frac{2i}{m}\omega-\frac{2j}{n}\omega'\right)$$

$$=(-1)^{rs}\prod_{i,j}\phi_{rs}v_{ij}\times\left\{\theta\left(\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right)\right\}^{mn}$$

$$\times\exp\left\{-mn\left(\frac{ra}{m}+\frac{sa'}{n}\right)\left(\frac{r\omega}{m}+\frac{s\omega'}{n}\right)+nr(a\omega-\beta)+ms(a'\omega'-\beta')\right\}$$

$$\div\prod_{i,j}e^{(ia/m+ja'/n)[v_{ij}-(2i\omega/m)-(2j\omega'/n)]}\theta\left(\frac{2i}{m}\omega+\frac{2j}{n}\omega'\right).$$

The part of this expression after the sign ÷ is common to all the terms, since it does *not contain* r or s; it may therefore be discarded

If, then, we write $\theta_{rs} v$ for

$$\phi_{rs} v \times \theta\left(\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right) \exp\left\{-\left(\frac{r\alpha}{m}+\frac{s\alpha'}{n}\right)\left(\frac{r\omega}{m}+\frac{s\omega'}{n}\right)+\frac{r}{m}(\alpha\omega-\beta)\right.$$
$$\left.+\frac{s}{n}(\alpha'\omega'-\beta')\right\},$$

that is, for

$$\exp\left\{-\left(\frac{r\alpha}{m}+\frac{s\alpha'}{n}\right)\left(v+\frac{r\omega}{m}+\frac{s\omega'}{n}\right)+\frac{r}{m}(\alpha\omega-\beta)+\frac{s}{n}(\alpha'\omega'-\beta')\right\}$$
$$\times\theta\left(v+\frac{2r\omega}{m}+\frac{2s\omega'}{n}\right),$$

we have, finally,

$$\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(-1)^{rs+rn(m-1)+sm(n-1)}\prod_{i=1}^{mn}\theta_{rs} v_i = 0,$$

if $v_1, v_2, \ldots, v_{mn}$ are mn arguments whose sum

$$\equiv (mn-1)(n\omega+m\omega') \quad (\text{mod } 2n\omega, 2m\omega').$$

Resolution into Factors.

Let $F_{mn}(v_1, v_2, \ldots, v_{mn})$ denote the expression

$$\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(-1)^{rs+rn(m-1)+sm(n-1)}\prod_{i=1}^{mn}\theta_{rs} v_i.$$

In certain cases this can be expressed as the product of factors. For instance,

$$\sum_{r=0}^{m-1}(-1)^r\prod_{i=1}^{m}\cos\left(v_i+\frac{r\pi}{m}\right)=\frac{m}{2^{m-1}}\cos\sum_{i=1}^{m}v_i,$$

there being in this case only one factor. Before discussing the other cases, it may be well to note that the constants α, β, α', β' are known to be connected by a further relation, which may be found as follows. The function $\theta u/\theta(-u)$ has neither pole nor zero within the parallelogram, and

$$\frac{\theta(u+2\omega)}{\theta(-u-2\omega)} \div \frac{\theta u}{\theta(-u)} = e^{2(\beta-\alpha\omega)},$$

$$\frac{\theta(u+2\omega')}{\theta(-u-2\omega')} \div \frac{\theta u}{\theta(-u)} = e^{2(\beta'-\alpha'\omega')}.$$

Hence $\dfrac{d}{du}\log\dfrac{\theta u}{\theta(-u)}$ has no pole within the parallelogram, and is doubly periodic; therefore it is a constant, say 2κ, and

$$\log \frac{\theta u}{\theta(-u)} = 2\kappa u + \lambda.$$

Now θu has a simple zero when $u = 0$, and therefore

$$\operatorname*{Lim}_{u=0} \frac{\theta u}{\theta(-u)} = -1,$$

or

$$e^{\lambda} = -1,$$

so that

$$\frac{\theta u}{\theta(-u)} = -e^{2\kappa u}.$$

Hence

$$e^{2\kappa\omega} = -\frac{\theta\omega}{\theta(-\omega)} = -e^{-a\omega+\beta},$$

and we may put

$$\beta = (a+2\kappa)\,\omega + \iota\pi\,;$$

in the same way,

$$\beta' = (a'+2\kappa)\,\omega' + \iota\pi.$$

Then the known result in the case $m = n = 2$ may be written

$$F_{22}(v_1, v_2, v_3, v_4) = -2e^{(a+a'-\kappa)v_1+\kappa(v_2+v_3+v_4)-(a+a'-4\kappa)(\omega+\omega')}$$

$$\times\theta\left(\frac{v_1+v_2+v_3+v_4}{2}-\omega-\omega'\right)\theta\left(\frac{v_1+v_2-v_3-v_4}{2}-\omega-\omega'\right)$$

$$\times\theta\left(\frac{v_1-v_2+v_3-v_4}{2}-\omega-\omega'\right)\theta\left(\frac{v_1-v_2-v_3+v_4}{2}-\omega-\omega'\right).$$

From this may be derived the expression in the case $m = 2$, $n = 1$, by writing ω' for v_3, $\omega+\omega'$ for v_4. Thus

$$\theta\omega'\,\theta(\omega+\omega')\,F_{21}(v_1, v_2)$$

$$= -2e^{(a+4\kappa)\omega'}\,\theta\frac{v_1+v_2-\omega}{2}\,\theta\frac{v_1+v_2+\omega}{2}\,\theta\left(\frac{v_1-v_2+\omega}{2}-\omega'\right)$$

$$\times\theta\left(\frac{v_2-v_1+\omega}{2}-\omega'\right).$$

In the case $m = 3$, $n = 1$, F_{31} can be expressed as the product of factors if ω'/ω has the special value $\frac{1}{3}(\rho-1)$, where $\rho = e^{\frac{2}{3}\iota\pi}$, so that

$$\omega' = \rho\,(\omega+\omega').$$

For, let $\theta_1 u$, $\theta_2 u$ now denote

$$e^{-\frac{1}{3}a(u+\frac{1}{3}\omega)+\frac{1}{3}(a\omega-\beta)}\,\theta\left(u+\tfrac{2}{3}\omega\right),$$

$$e^{-\frac{2}{3}a(u+\frac{2}{3}\omega)+\frac{2}{3}(a\omega-\beta)}\,\theta\left(u+\tfrac{4}{3}\omega\right).$$

Then the *functions* $\theta_1 u/\theta u$, $\theta_2 u/\theta u$ have the periods 2ω, $6\omega'$, that is,

2ω, $2\rho\omega$, $2\rho^2\omega$; their poles are 0, $\pm 2\omega'$ or 0, $\pm\frac{2}{3}(\rho-1)\omega$. The zeros of $\theta_1 u/\theta u$ are $-\frac{2}{3}\omega$, $-\frac{2}{3}\omega\pm\frac{2}{3}(\rho-1)\omega$, that is $-\frac{2}{3}\omega$, $-\frac{2}{3}\rho\omega$, $-\frac{2}{3}\rho^2\omega$, and those of $\theta_2 u/\theta u$ are $\frac{2}{3}\omega$, $\frac{2}{3}\omega\pm\frac{2}{3}(\rho-1)\omega$, that is $\frac{2}{3}\omega$, $\frac{2}{3}\rho\omega$, $\frac{2}{3}\rho^2\omega$. Hence $\theta_1 u/\theta u$ and $\theta_1\rho u/\theta\rho u$ are functions of u having the same periods, poles and zeros; this is evident in the case of the periods and zeros; a pole of $\theta_1\rho u/\theta\rho u$ is $\frac{2}{3}(\rho-1)\rho^2\omega$, which is equal to $\frac{2}{3}(\rho-1)\omega+2\omega$, and therefore equivalent to a pole of $\theta_1 u/\theta u$; similarly for the pole $-\frac{2}{3}(\rho-1)\rho^2\omega$. Thus $\theta_1\rho u/\theta\rho u$ only differs from $\theta_1 u/\theta u$ by a constant factor which is found to be ρ^2 on giving u a small value. Thus

$$\frac{\theta_1\rho u}{\theta_1 u}=\frac{\theta\rho u}{\rho\theta u},=\frac{\theta_2\rho u}{\theta_2 u}\text{ similarly.}$$

Each of the fractions in fact

$$=\exp\left\{\frac{u^2}{4\omega'}(\alpha'\rho^2-\alpha'\rho-\alpha\rho)+\kappa u(\rho-1)\right\}.$$

For let $\quad\dfrac{\theta\rho u}{\rho\theta u}\exp\left\{\dfrac{u^2}{4\omega'}(\alpha'\rho-\alpha'\rho^2+\alpha\rho)+\kappa u(1-\rho)\right\}=\chi(u).$

Then

$$\frac{\chi(u+2\omega)}{\chi u}=\frac{\theta(\rho u+6\omega'+2\omega)}{\theta\rho u}\;\frac{\theta u}{\theta(u+2\omega)}$$

$$\times\exp\left\{\frac{\omega}{\omega'}(u+\omega)(\alpha'\rho-\alpha'\rho^2+\alpha\rho)+2\kappa\omega(1-\rho)\right\}$$

$$=\exp\{(3\alpha'+\alpha)\rho u+6\alpha'\omega'+6\alpha'\omega+3\beta'+\beta-\alpha u-\beta$$
$$+(\rho^2-1)(u+\omega)(\alpha'\rho-\alpha'\rho^2+\alpha\rho)+2\kappa\omega(1-\rho)\}=1;$$

also

$$\frac{\chi(u+2\omega')}{\chi u}=\frac{\theta(\rho u-2\omega-4\omega')}{\theta\rho u}\;\frac{\theta u}{\theta(u+2\omega')}$$

$$\times\exp\{(u+\omega')(\alpha'\rho-\alpha'\rho^2+\alpha\rho)+2\kappa\omega'(1-\rho)\}$$

$$=\exp\{-\alpha'(2\rho u-4\omega-6\omega')-2\beta'-\alpha(\rho u-2\omega)-\beta-\alpha'u-\beta'$$
$$+(u+\omega')(\alpha'\rho-\alpha'\rho^2+\alpha\rho)+2\kappa\omega'(1-\rho)\}=1.$$

These reductions depend on the relations

$$3\omega'+\omega=\rho\omega,\quad\rho\omega'=-\omega-2\omega',\quad\alpha\omega'-\alpha'\omega=\iota\pi.$$

It follows that χu is doubly periodic, and therefore constant, since it has no poles. Its value is 1 when $u=0$, and hence the statement made is proved.

Now $F_{31}(v_1, v_2, v_3)=\theta v_1\theta v_2\theta v_3+\theta_1 v_1.\theta_1 v_2.\theta_1 v_3+\theta_2 v_1.\theta_2 v_2.\theta_2 v_3.$

Hence $F_{31}(v_1, v_2, v_3)$, $F_{31}(v_1, \rho v_2, \rho^2 v_3)$, $F_{31}(v_1, \rho^2 v_2, \rho v_3)$ only differ by an exponential factor, and the zeros of each, as a function of v_1, are (mod 2ω, $6\omega'$), $-v_2-v_3$, $-\rho v_2-\rho^2 v_3$, $-\rho^2 v_2-\rho v_3$. Now

$$2\omega = -(\rho-\rho^2)(2\omega+4\omega'),$$
$$6\omega' = (\rho-\rho^2)(4\omega+6\omega'),$$

so that the periods 2ω, $6\omega'$ are equivalent to $(\rho-\rho^2)\,2\omega$, $(\rho-\rho^2)\,2\omega'$, and the zeros and periods of $F_{31}(v_1, v_2, v_3)$ coincide with those of

$$\theta\frac{v_1+v_2+v_3}{\rho-\rho^2}\,\theta\frac{v_1+\rho v_2+\rho^2 v_3}{\rho-\rho^2}\,\theta\frac{v_1+\rho^2 v_2+\rho v_3}{\rho-\rho^2}.$$

In fact, if

$$f(v_1, v_2, v_3) = \theta\frac{v_1+v_2+v_3}{\rho-\rho^2}\,\theta\frac{v_1+\rho v_2+\rho^2 v_3}{\rho-\rho^2}\,\theta\frac{v_1+\rho^2 v_2+\rho v_3}{\rho-\rho^2}$$
$$\times(\rho-\rho^2)\exp\Big[-\frac{1}{4\omega}\{2v_1^2+(v_2+v_3)^2\}\{\rho a+a'(\rho-\rho^2)\}$$
$$-\kappa(2\rho^2 v_1-v_2-v_3)\Big]$$

it may be verified without difficulty that

$$F_{31}(v_1, v_2, v_3) \div f(v_1, v_2, v_3)$$

is unchanged by the following substitutions for v_1, v_2, v_3, namely, $v_1+2\omega, v_2, v_3$; $v_1+6\omega', v_2, v_3$; $v_1+2\omega, v_2+2\omega, v_3+2\omega$; $v_1+2\omega', v_2+2\omega', v_3+2\omega'$; $v_1, v_2+2\omega, v_3-2\omega$; $v_1, v_2+2\omega', v_3-2\omega'$. Hence $F_{31}\div f$ is a doubly periodic function of each of the three arguments, and it is never infinite. Thus it is a constant. Again, if $v_2 = -v_3 = \frac{2}{3}\omega$, this constant is readily seen to be 1. Thus $F_{31} = f$, that is, when

$$\omega+\omega' = \rho^2\omega,$$

$$F_{31}(v_1, v_2, v_3) = (\rho-\rho^2)\exp\Big[-\frac{1}{4\omega}\{2v_1^2+(v_2+v_3)^2\}\{\rho a+(\rho-\rho^2)\,a'\}$$
$$-\kappa(2\rho^2 v_1-v_2-v_3)\Big]$$
$$\times\,\theta\frac{v_1+v_2+v_3}{\rho-\rho^2}\,\theta\frac{v_1+\rho v_2+\rho^2 v_3}{\rho-\rho^2}\,\theta\frac{v_1+\rho^2 v_2+\rho v_3}{\rho-\rho^2}.$$

This result is simplified by taking θu to be σu, for then $\kappa = 0$, $a = a'(\rho-1)$, and $F_{31}(v_1, v_2, v_3)$, that is

$$\sigma v_1\sigma v_2\sigma v_3-e^{-\frac{2}{3}\eta(v_1+v_2+v_3+\omega)}\,\sigma(v_1+\tfrac{2}{3}\omega)\,\sigma(v_2+\tfrac{2}{3}\omega)\,\sigma(v_3+\tfrac{2}{3}\omega)$$
$$+e^{-\frac{4}{3}\eta(v_1+v_2+v_3+2\omega)}\,\sigma(v_1+\tfrac{4}{3}\omega)\,\sigma(v_2+\tfrac{4}{3}\omega)\,\sigma(v_3+\tfrac{4}{3}\omega)$$
$$=(\rho-\rho^2)\,\sigma\frac{v_1+v_2+v_3}{\rho-\rho^2}\,\sigma\frac{v_1+\rho v_2+\rho^2 v_3}{\rho-\rho^2}\,\sigma\frac{v_1+\rho^2 v_2+\rho v_3}{\rho-\rho^2}.$$

On Multiply Infinite Series and on an Extension of Taylor's Series. *By* M. G. MITTAG-LEFFLER. Communicated June 8th, 1899.*

I have introduced in a former publication† a new geometrical conception, the *star*-figure.

In the plane of the complex variable x let an area be generated in the following manner:—Round a fixed point a let a vector l (a straight line terminated at a) revolve once. On each position of the vector determine uniquely a point, say a_l, at a distance from a greater than a given positive quantity, this quantity being the same for all positions of the vector. The points thus determined may be at a finite or at an infinite distance from a. When the distance between a_l and a is finite, the part of the vector from a_l to infinity is excluded from the plane of the variable.

The *star* is the region which remains when all these sections (*coupures*) in the plane of x have been made. The fixed point a is called the *centre* of the star. It is convenient to name the points a_l the *summits* of the star, and to introduce the following definition:—One star is *inscribed* in another, if all the points of the first star belong to the second and if the two stars have common summits; the second star is then *circumscribed* about the first.

By the use of the well known notion of a *limiting value*, the results which were obtained in the paper referred to can be stated as follows:—

Let $F(a),\ F^{(1)}(a),\ \dots,\ F^{(\nu)}(a),\ \dots$

be an array of quantities satisfying Cauchy's condition;‡ *denote by*

$$G_n(x \mid a)$$

the polynomial in x,

$$\sum_{h_1=0}^{n^2} \sum_{h_2=0}^{n^4} \dots \sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\,h_2!\,\dots\,h_n!}\,F^{(h_1+h_2+\dots+h_n)}(a)\left(\frac{x-a}{n}\right)^{h_1+h_2+\dots+h_n},$$

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* Received, in French, May 4th, 1900. The English translation, made by Mr. R. W. H T. Hudson, received from Prof. Mittag-Leffler, June 19th, 1900.

† "Sur la représentation analytique d'une branche uniforme d'une fonction *monogène* (première note)," *Acta Mathematica*, tome XXIII., pp. 43–62.

‡ *Acta Mathematica*, *l.c.*, pp. 43, 44.

which is formed by means of these quantities. Consider the limiting value

$$\operatorname*{Lim}_{n=\infty} G_n(x \mid a).$$

There exists a star A, with centre a, which is uniquely determined when the quantities $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... are once given, and which possesses the following properties in relation to the limiting value $\operatorname*{Lim}_{n=\infty} G_n(x \mid a)$:—

This expression is uniformly convergent for every domain in the interior of A, but is not uniformly convergent for any two-dimensional continuum which includes a summit of A. It defines, within A, the branch $FA(x)$ of a monogenic function. This branch is regular within A, but has the summits of A for critical points.

Further, it possesses the property

$$\left(\frac{d^\mu FA(x)}{dx^\mu}\right)_{x=a} = F^{(\mu)}(a) \quad (\mu = 0, 1, 2, \ldots).$$

The functional branch $FA(x)$ can be defined, not only by $G_n(x \mid a)$, but also by an infinite number of other polynomials

$$g_n(x \mid a) = \sum_\nu c_\nu^{(n)} F^{(\nu)}(a)(x-a)^\nu,$$

in which each of the coefficients $c_\nu^{(n)}$ is a numerical quantity, given in terms of ν and n, and independent of a, of $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ..., and of x, and which possesses in relation to A the same properties as $G_n(x \mid a)$.

Among the stars which can be inscribed in the star A is to be specially noticed the circle C, centre a, the circumference of which passes through the summit of A nearest to the centre, in accordance with our definition. Corresponding to this circle we have the limiting value

$$\operatorname*{Lim}_{n=\infty} \sum_{\nu=0}^{n} \frac{1}{\nu!} F^{(\nu)}(a)(x-a)^\nu,$$

which is known as *Taylor's series*. Taylor's series possesses in relation to the circle C the same properties which I have just stated for

$$\operatorname*{Lim}_{n=\infty} \sum_{h_1=0}^{n^2} \sum_{h_2=0}^{n^4} \ldots \sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\, h_2! \ldots h_n!} F^{(h_1+h_2+\ldots+h_n)}(a) \left(\frac{x-a}{n}\right)^{h_1+h_2+\ldots+h_n},$$

in relation to A, with one very important exception. Every summit of A is a singular point of the functional branch $FA(x)$. This is not the case for C with respect to $FC(x)$. It can be asserted only that there exists at least one summit of C (*i.e.*, a point on the circumference of C) which is a singular point for the branch $FC(x)$.

But, on the other hand, Taylor's series possesses in relation to C the property, which cannot in general be attributed to the limiting value

$$\operatorname*{Lim}_{n=\infty} \sum_{h_1=0}^{n^2} \sum_{h_2=0}^{n^4} \dots \sum_{h_n=0}^{n^{2n}} \frac{1}{h_1!\, h_2! \dots h_n!} F^{(h_1+h_2+\dots+h_n)}(a) \left(\frac{x-a}{n}\right)^{h_1+h_2+\dots+h_n},$$

in relation to A, that it does not converge for any point outside C.

Now between C and A there exist intermediary stars K, unlimited in number, each of which in succession circumscribes the preceding; with each star corresponds a limiting value possessing in relation to it the same properties which I have stated for Taylor's series in relation to C. These new limiting values have further the property that they include Taylor's series as a special case.

There are different classes of expressions which fulfil these conditions. My intention is to investigate in this note one of these classes, as follows:—

Let
$$f_{h_1 h_2 \dots h_n} \left\{ \begin{matrix} h_1 = 0, 1, 2, \dots, \infty \\ h_2 = 0, 1, 2, \dots, \infty \\ \dots \quad \dots \quad \dots \\ h_n = 0, 1, 2, \dots, \infty \end{matrix} \right\}$$

be an n-ply infinite array of functions of a certain number of variables.

The multiple series $\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \dots \sum_{h_n=0}^{\infty} f_{h_1 h_2 \dots h_n}$

is generally defined as being equal to a single series, the terms of which are the different functions $f_{h_1 h_2 \dots h_n}$ made to correspond with the integers 0, 1, 2, ..., according to a certain law.

It is, however, useful to study this series from another point of view, which, in spite of its great simplicity and the advantage to be obtained from it, does not appear so far to have attracted the attention of mathematicians.*

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* See "Om den analytiska framställningen, &c., Första meddelande, 11 Maj *1898*," *Vet. Ak. Öfversicht.*

I make the following definition:—

I suppose that the series

$$f_{h_1\ldots h_{n-1}} = \sum_{h_n=0}^{\infty} f_{h_1\ldots h_n},$$

$$f_{h_1\ldots h_{n-2}} = \sum_{h_{n-1}=}^{\infty} f_{h_1\ldots h_{n-1}},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$f_{h_1} = \sum_{h_2=0}^{\infty} f_{h_1h_2},$$

$$f = \sum_{h_1=0}^{\infty} f_{h_1}$$

are all convergent for a certain value of the variables.

I say then that *the series*

$$f = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \ldots \sum_{h_n=0}^{\infty} f_{h_1h_2\ldots h_n}$$

is an n-ply infinite series which is convergent for this value.

I suppose further that the series

$$f_{h_1h_2\ldots h_{n-1}},\ f_{h_1h_2\ldots h_{n-2}},\ \ldots,\ f_{h_1},\ f$$

are all uniformly convergent in a common domain B within the domain of existence of the functions $f_{h_1\ldots h_n}$.

I say then that the series is *an n-ply infinite series which is uniformly convergent in the domain B.* Having laid down this definition, I can enunciate the following theorem:—*

The functions $\qquad f_{h_1h_2\ldots h_n}(x_1 \ldots x_m) \quad \left\{\begin{matrix} h_1 = 0,\ 1,\ 2,\ \ldots,\ \infty \\ h_2 = 0,\ 1,\ 2,\ \ldots,\ \infty \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ h_n = 0,\ 1,\ 2,\ \ldots,\ \infty \end{matrix}\right\}$

being analytic functions of the variables $x_1, \ldots, x_m$, *uniform and regular in the domain B, and the series*

$$f = \sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \ldots \sum_{h_n=0}^{\infty} f_{h_1h_2\ldots h_n}$$

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* This theorem is an immediate consequence of the theorem demonstrated by Weierstrass in his *memoir* "*Zur* Functionenlehre," § 2, *Werke*, Bd. II., pp. 205–208.

being an n-ply infinite series, which is uniformly convergent in the domain B, then the series represents, in the same domain B, a uniform regular function of the variables $x_1, \ldots, x_m$.

It is easy to see that the conception of an n-ply infinite series is very different from the generally received notion of a multiple series, and that the region of convergence of the first series is generally of much greater extent than that of the second.

Take as an example

$$f(x) = \frac{1}{1-x}.$$

Denoting by ζ a point on the real axis between 0 and -1, we have

$$f(x) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} f^{(\nu)}(\zeta)\,(x-\zeta)^{\nu}.$$

This equality holds for $|x-\zeta| \leqslant |\zeta|$.

Then $$f(2\zeta) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} f^{(\nu)}(\zeta)\,\zeta^{\nu} = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu}.$$

If we regard the series

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu}$$

as a double series in the ordinary sense, we shall have

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu} = \sum_{h=0}^{\infty} \frac{1}{h!} f^{(h)}(0)\,(2\zeta)^{h},$$

and this series will be then convergent only for $0 \geqslant \zeta > -\frac{1}{2}$. If, on the contrary, the series

$$\sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu!\,\mu!} f^{(\nu+\mu)}(0)\,\zeta^{\nu+\mu}$$

is regarded as a doubly infinite series, the convergence holds for $0 > \zeta > -1$.

It is by an application of my conception of an n-ply infinite series in the case where the functions $f_{h_1 h_2 \ldots h_n}$ involve only one variable x that I obtain a class of limiting values possessing the properties *which I have stated* above.

We have, in fact, the following theorem :—

THEOREM.—*Let* $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... *be any quantities satisfying Cauchy's condition, and let*

$$\sum_{h_1=0}^{\infty}\sum_{h_2=0}^{\infty}\dots\sum_{h_n=0}^{\infty} c_{h_1 h_2 \dots h_n} F^{(h_1+h_2+\dots+h_n)}(a)\,(x-a)^{h_1+h_2+\dots+h_n}$$

be an n-ply infinite series in which $c_{h_1 h_2 \dots h_n}$ *denote certain numerical constants, independent of the quantities* $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... *as well as of a and of x.*

It is always possible to choose these constants in such a way that the series may possess the following properties :—

It will have a star of convergence $A^{(1/n)}$, *such that the series is uniformly convergent for every domain within* $A^{(1/n)}$, *but converges nowhere outside* $A^{(1/n)}$.

This star $A^{(1/n)}$ *is inscribed in the star A of the elements** $F(a)$, $F^{(1)}(a)$, ..., $F^{(n)}(a)$, ..., *and when* $n \geqq \bar{n}$, *where the positive integer* $\bar{n}$ *is taken sufficiently great, it contains within itself any finite domain which is contained in A.*

Further, the star $A^{(1/n)}$ *is inscribed in the star* $A^{(1/n')}$, *when* $n < n'$.

The equation

$$FA^{(1/n)}(x) = \sum_{h_1=0}^{\infty}\sum_{h_2=0}^{\infty}\dots\sum_{h_n=0}^{\infty} c_{h_1 h_2 \dots h_n} F^{(h_1+h_2+\dots+h_n)}(a)\,(x-a)^{h_1+h_2+\dots+h_n}$$

holds throughout the interior of $A^{(1/n)}$. *For* $n = 1$ *the series becomes Taylor's series.*

Among the different modes of fixing $c_{h_1 h_2 \dots h_n}$, I will bring forward one, with which corresponds a star of convergence $A^{(1/n)}$, obtained by the following simple construction:—Select one vector l through the centre a of the star A. Construct a system of circles having their centres a, η_1, η_2, ..., η_{n-1} on l, so that each passes through the centre of the preceding. I shall denote the radii by r, r_1, r_2, ..., r_{n-1}. The centres η_1, η_2, ..., η_{n-1} are chosen in such a manner that each circle cuts the preceding in the points of contact of tangents from a to the former, and that $|\eta_1 - a| = r_1 = r$. It is evident that, if the radius r be taken sufficiently small, this system of circles will be included in A. The star $A^{(1/n)}$ is obtained by measuring along l the length

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* *I.e.*, the star belonging to the elements $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... See "Sur la représentation analytique d'une branche uniforme d'une fonction monogène (première note)," *Acta Mathematica*, tome XXIII., p. 48.

$| \eta_{n-1} - a | + r_{n-1}$, then substituting for r its superior limit ρ and making l revolve once round a. With this form of star of convergence, the coefficients $c_{h_1 h_2 \dots h_n}$ are

$$c_{h_1 h_2 \dots h_n} = \left(\frac{\sin^2 a_{n-1}}{1+\sin a_{n-1}}\right)^{h_1+h_2+\dots+h_{n-1}} \frac{\left(\dfrac{1}{\sin a_{n-1}}\right)^{h_1+h_2}}{h_1!\, h_2!} \; \frac{\left(\dfrac{2}{\sin a_{n-2}}\right)^{h_3}}{h_3!}$$

$$\times \frac{\left(\dfrac{2^2}{\sin a_{n-3}}\right)^{h_4}}{h_4!} \dots \frac{\left(\dfrac{2^{\nu-2}}{\sin a_{n+1-\nu}}\right)^{h_\nu}}{h_\nu!} \dots \frac{(2^{n-2})^{h_{n-1}+h_n}}{h_{n-1}!\, h_n!}$$

$$(n = 2, 3, \dots),$$

$$\sin a_{\mu+1} = -\tfrac{1}{4}\sin^2 a_\mu + \tfrac{1}{4}\sin a_\mu \sqrt{8+\sin^2 a_\mu} \quad (\mu = 1, 2, 3, \dots),$$

$$\sin a_1 = 1,$$

$$c_{h_1} = \frac{1}{h_1!}.$$

The root which occurs in the expression for $\sin a_{\mu+1}$ is the positive root.

We have then

$$c_{h_1} = \frac{1}{h_1!},$$

$$c_{h_1 h_2} = \frac{1}{h_1!\, h_2!},$$

$$c_{h_1 h_2 h_3} = \frac{1}{h_1!\, h_2!\, h_3!},$$

$$c_{h_1 h_2 h_3 h_4} = \frac{1}{h_1!\, h_2!\, h_3!\, h_4!} \left(\frac{\sin a_3}{1+\sin a_3}\right)^{h_1+h_2} \left(\frac{4\sin^2 a_3}{1+\sin a_3}\right)^{h_3+h_4}$$

$$\left(\sin a_3 = \frac{\sqrt{33}-1}{16}\right),$$

... ...

Since the n-ply infinite series

$$\sum_{h_1=0}^{\infty} \sum_{h_2=0}^{\infty} \dots \sum_{h_n=0}^{\infty} \frac{1}{h_1!\, h_2! \dots h_n!} F^{(h_1+h_2+\dots+h_n)}(a)\, (x-a)^{h_1+h_2+\dots+h_n}$$

has a star of convergence in the same sense as Taylor's series has, when $n = 1, 2, 3$, the question arises whether the same is true for $n > 3$. It is not so, if the elements $F(a)$, $F^{(1)}(a)$, ..., $F^{(\mu)}(a)$, ... are chosen arbitrarily. It may happen then that the series converge at a point $x = x'$, without converging at $x = ax'$, a being a real positive quantity less than unity.

This is a new example of the danger in analysis of drawing general *conclusions from special* cases.

Canonical Reduction of Linear Substitutions and Bilinear Forms, with a Dynamical Application. By T. J. I'A. BROMWICH.

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The following paper falls into three parts: in the first we solve the problem of reducing to a canonical form the generalized linear substitution consisting of n equations between $2n$ quantities: the second part applies the results of the first to a problem in dynamics—that of finding the small oscillations of a system about a state of steady motion by Hamilton's equations; the third part solves the problem of the simultaneous reduction of a pair of bilinear forms.

Owing to the fact that the first and third parts were written independently, there are some differences of notation; but I hope that these are not sufficient to cause any confusion.

1. *Reduction of a Generalized Linear Substitution to a Canonical Form.*

In a note presented to the Society (November, 1899) I gave an investigation of a method of reducing a linear substitution to its canonical form; in what follows I have investigated a similar method for reducing a set of n linear equations between n ξ's and n x's.

It will be seen that my method was suggested by Darboux's reduction of two quadratic forms,[*] which was extended to bilinear forms by Stickelberger.[†] In fact, the forms for the reduced coordinates will include Stickelberger's as particular cases. We might apply Muth's method[‡] and deduce the reduction of the linear substitution from that of two simultaneously reduced bilinear forms; but it appears to be easier to reduce the substitution independently, owing to the fact that only one set of n variables has to be reduced.

In my reduction I have considered specially two cases of some importance; namely when $| B | = 0$ and $| A - \lambda B | \equiv 0$, in the notation of the following work. The need of these special investigations was pointed out by Prof. Burnside, in connexion with the note referred to above.

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[*] *Journal de Mathématiques* (*Liouville*, 2^{me} série), 1874, t. XIX., pp. 347–396.

[†] *Crelle's Journal für die Mathematik* (1879), Bd. LXXXVI., pp. 20–43.

[‡] *Theorie und Anwendung der Elementartheiler* (Leipzig, 1899), Arts. 79, 80.

Suppose that we are given the generalized linear substitution

$$\begin{aligned} a_{11}x_1+\ldots+a_{1n}x_n &= b_{11}\xi_1+\ldots+b_{1n}\xi_n, \\ \ldots \quad \ldots \quad &= \quad \ldots \quad \ldots \\ a_{n1}x_1+\ldots+a_{nn}x_n &= b_{n1}\xi_1+\ldots+b_{nn}\xi_n. \end{aligned}$$

Then, whatever $w_1, \ldots, w_n$ may be

$$\Sigma a_{rs}w_rx_s = \Sigma b_{rs}w_r\xi_s \quad (r, s = 1, 2, \ldots, n),$$

and we shall write

$$\begin{aligned} A &= A(w, x) = \Sigma a_{rs}w_rx_s, \\ B &= B(w, x) = \Sigma b_{rs}w_rx_s, \\ B(\xi) &= B(w, \xi) = \Sigma b_{rs}w_r\xi_s, \end{aligned}$$

so that the equations above yield

$$A = B(\xi).^*$$

In order to reduce the substitution, we must in the first place divide the determinant $|A-\lambda B|$ into its invariant-factors (*Elementartheiler*); to do this we divide $|A-\lambda B|$ by the H.C.F. of all its first minors; the H.C.F. of all the first minors by that of all the second minors, and so on.

We suppose that $(\lambda-c)$ is a factor of $|A-\lambda B|$, and let $e_1, e_2, \ldots, e_k, \ldots$ be the indices of $(\lambda-c)$ in the first, second, ..., k^{th}, ... quotients as just constructed. Then the corresponding powers of $(\lambda-c)$ are the invariant-factors which belong to $(\lambda-c)$.

Construct the form $(A-\lambda B)^{-1}$ which is reciprocal to $(A-\lambda B)$; this can obviously be expanded in a power-series of $(\lambda-c)$, and the

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* It may be useful to point out here, once for all, that in what follows we shall suppose the symbolical product of two bilinear forms in w, x to be given by

$$AB = \Sigma \frac{\partial A}{\partial x} \frac{\partial B}{\partial w};$$

or, what is the same thing, by substituting for each x in A the corresponding $\frac{\partial B}{\partial w}$; *or, again, by substituting* $\frac{\partial A}{\partial x}$ for each w in B.

first term in the expansion will be $(\lambda - c)^{-e_1}$.* Or we may write

$$(A-\lambda B)^{-1} = Dt^{-e} + P(t) \quad (e = e_1),$$

$$t = \lambda - c,$$

where D is a bilinear form which contains no powers of t beyond t^{e-1}; and $P(t)$ is a bilinear form which contains no negative powers of t. But we have

$$E = (A-\lambda B)(A-\lambda B)^{-1} = (A-\lambda B)^{-1}(A-\lambda B),$$

where E is the unit-form (*Einheitsform*) or

$$E = w_1 x_1 + \ldots + w_n x_n.$$

Hence we have

$$Et^e = D\left[A-(c+t)B\right] + t^e P(t)\left[A-(c+t)B\right]$$

Now write $\qquad DB = y_1 + y_2 t + \ldots + y_e t^{e-1};$

then $\qquad DA = DB(\xi) \quad$ (by the relation found on p. 80)

$$= \eta_1 + \eta_2 t + \ldots + \eta_e t^{e-1},$$

where η_r is the same function of the ξ's that y_r is of the x's.

Our equations now yield, if

$$P_1(t) = P(t)\left[A-(c+t)B\right],$$

$$(\eta_1 + \eta_2 t + \ldots + \eta_e t^{e-1}) - (c+t)(y_1 + y_2 t + \ldots + y_e t^{e-1}) = t^e\left[P_1(t) - E\right];$$

or, equating coefficients of powers of t,

$$\eta_1 = cy_1, \quad \eta_2 = cy_2 + y_1, \quad \eta_r = cy_r + y_{r-1} \quad (r = 2, 3, \ldots, e_1).$$

Hitherto we have regarded the symbols involved as bilinear forms; but, if $w_1, \ldots, w_n$ be now considered as constants, the quantities $\eta_1, \eta_2, \ldots, y_1, y_2, \ldots$ will be the same linear functions of the ξ's and x's respectively; thus the given equations have been reduced to a simple set of equations in the manner prescribed.

It may be convenient, for the purpose of reference, to have the values of the y's written out in a complete form; to do this we note that $t^{-e}DB$ is equal to the first e terms of $\left[A-(c+t)B\right]^{-1}B$; or to the first e terms in the result of replacing the x's in $\left[A-(c+t)B\right]^{-1}$ by $\frac{\partial B}{\partial w}$'s. Thus

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* See, for example, Muth's *Elementartheiler*, p. 135; the fact is, however, almost intuitive, on examining the construction of $(A-\lambda B)^{-1}$.

$(y_1+y_2t+\ldots+y_et^{e-1})\,t^{-e}+$ terms without negative powers of t

$$= -\frac{1}{\Delta_0}\begin{vmatrix} a_{11}-(c+t)\,b_{11}, & \ldots, & a_{1n}-(c+t)\,b_{1n}, & (b_{11}x_1+\ldots+b_{1n}x_n) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}-(c+t)\,b_{n1}, & \ldots, & a_{nn}-(c+t)\,b_{nn}, & (b_{n1}x_1+\ldots+b_{nn}x_n) \\ w_1, & \ldots, & w_n, & 0 \end{vmatrix},$$

in which $$\Delta_0 = |\, A-(c+t)\,B \,|.$$

The y's may be slightly generalized by taking

$$Y_1+Y_2t+\ldots+Y_et^{e-1}+\ldots = Q(t)\,(y_1+y_2t+\ldots+y_et^{e-1}),$$

where $Q(t)$ is any power-series of t with constant coefficients which starts with a constant term.

Referring to previously known reductions, it may be mentioned that Stickelberger's Y_1, Y_2, ...,* are the same as these. If $B=E$, the Y's are the same as the z's found in a different form by Prof. Burnside (*l.c. supra*); and they have been given in my note for the special case when one w is unity, the remaining w's being zero. Mr. H. F. Baker† has shown that Prof. Burnside's reduction is given by the following:—

$$\Sigma z'^{(s)}t^s = \chi(A,\, c+t),$$

and that $$\chi(A,\, c+t) = \psi(c+t)\left[A-(c+t)\,E\right]^{-1},$$

where $\psi(c+t)$ is the quotient of $|\, A-(c+t)\,E \,|$ by the H.C.F. of all its first minors. This shows that Prof. Burnside's z's must be contained in the Y's above.

In order to proceed to find other reduced equations, we introduce the subsidiary form

$$\begin{aligned} X_k = {} & x_{n+1}\,(p_1'w_1+\ldots+p_n'w_n)+\ldots+x_{n+k}\,(p_1^kw_1+\ldots+p_n^kw_n) \\ & +w_{n+1}\,(q_1'x_1+\ldots+q_n'x_n)+\ldots+w_{n+k}\,(q_1^kx_1+\ldots+q_n^kx_n), \end{aligned}$$

and in this form the p's and q's are quite arbitrary, the indices acting as additional suffixes and not as exponents. Further $x_{n+1}, \ldots, x_{n-k}$, $w_{n+1}, \ldots, w_{n+k}$ are only to be used in performing the symbolical multiplications which are required; after the products are formed the extra x's and w's are all to be put zero.

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* *Crelle* (1879), Bd. LXXXVI., p. 26, equation (7).

† *Proc. Lond. Math. Soc.* (1899), Vol. XXX., pp. 195–198; see also Frobenius, *Berliner Sitzungsberichte*, 1896, pp. 606 *et seq.*

If we form the product $\left[A+X_k-(c+t)B\right]^{-1}X_k$ (on this hypothesis), it is easily seen to be

$$-\frac{1}{\Delta_k}\begin{vmatrix} d_{11}, & \ldots, & d_{1n}, & p'_1, & \ldots, & p^k_1, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{n1}, & \ldots, & d_{nn}, & p'_n, & \ldots, & p^k_n, & 0 \\ q'_1, & \ldots, & q'_n, & 0, & \ldots, & 0, & \dfrac{\partial X_k}{\partial w_{n+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ q^k_1, & \ldots, & q^k_n, & 0, & \ldots, & 0, & \dfrac{\partial X_k}{\partial w_{n+k}} \\ w_1, & \ldots, & w_n, & 0, & \ldots, & 0, & 0 \end{vmatrix},$$

where
$$\Delta_k = \left| A+X_k-(c+t)B \right|,$$
$$d_{rs} = a_{rs}-(c+t)b_{rs}.\text{[*]}$$

On expanding this form we see that each term contains no negative powers of t. Thus $\left[A+X_k-(c+t)B\right]^{-1}\left[A-(c+t)B\right]$ can be expanded in powers of t, without negative powers.

Now we have, as before,

$$\left[A+X_k-(c+t)B\right]^{-1} = Ft^{-e}+P(t) \quad (e = e_{k+1}).$$

where F is a bilinear form containing no powers of t beyond t^{e-1}; and $P(t)$ contains no negative powers of t.

Hence we find that

$$F\left[A-(c+t)B\right] = t^eR(t),$$

where $R(t)$ contains no negative powers of t. Let then

$$FB = y_1+y_2t+\ldots+y_et^{e-1},$$
$$FA = FB(\xi) \quad \text{(by the original equations)}$$
$$= \eta_1+\eta_2t+\ldots+\eta_et^{e-1},$$

and thus we find by equating coefficients of powers of t,

$$\eta_1 = y_1, \quad \eta_2 = y_2+cy_1, \quad \eta_r = y_r+cy_{r-1} \quad (r = 2, 3, \ldots, e_{k+1}).$$

We may generalize the y's by multiplying FB by a power-series with any constant coefficients, the series containing no negative powers of t.

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[*] The p's and q's are determined so that Δ_k is not divisible by a higher power of t than *any* k^{th} minor of Δ_0; *cf.* Stickelberger's paper quoted on the last page.

The values of the y's will be the coefficients of the first e_{k+1} terms in the expansion of

$$-\frac{1}{\Delta_k}\begin{vmatrix} d_{11}, & \dots, & d_{1n}, & p'_1, & \dots, & p^k_1, & (b_{11}x_1+\dots+b_{1n}x_n) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ d_{n1}, & \dots, & d_{nn}, & p'_n, & \dots, & p^k_n, & (b_{n1}x_1+\dots+b_{nn}x_n) \\ q'_1, & \dots, & q'_n, & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q^k_1, & \dots, & q^k_n, & 0, & \dots, & 0, & 0 \\ w_1, & \dots, & w_n, & 0, & \dots, & 0, & 0 \end{vmatrix}$$

in powers of t.

This agrees with Stickelberger's result; but it may be noted that the p's and q's need not be the same in finding successive sets of y's for the reduction of a linear substitution; while in reducing a pair of bilinear forms it is necessary that the constants $p'_1, \dots, p'_n$ should be the same in the last set of y's as in the first.

Proceeding thus for $k = 0, 1, 2, \dots$, we get in all $(e_1+e_2+e_3+\dots)$ y's which correspond to the factor $(\lambda-c)$. But $e_1+e_2+\dots = h$ is the index of $(\lambda-c)$ in $|A-\lambda B|$. Hence the total number of y's found from all the factors of $|A-\lambda B|$ will be Σh or n; and this ought to be the case.

If $|B| = 0$ (and $|A-\lambda B| \not\equiv 0$ for all values of λ), then the equation

$$|A-\lambda B| = 0$$

will have one or more roots infinite. To be perfectly general we must examine the effect of such roots, although the method above requires a very slight modification to apply.

Let us suppose that h_k is the index of the highest power of λ which appears in *any* one of the k^{th} minors; then $(n-k-h_k)$ roots of every k^{th} minor will be infinite, for the degree in λ ought to be $(n-k)$. Thus here we have to take

$$e_k = \left[n-(k-1)-h_{k-1}\right]-\left[n-k-h_k\right]$$

$$= 1+h_k-h_{k-1}.$$

Clearly $$e_{k+1} = 0,$$

if $$h_k = n-k.$$

We now have, writing $\lambda = 1/t$, and expanding in powers of t,

$$(A+X_k-B/t)^{-1} = Ft^{1-e}+tP(t) \quad (e = e_k),$$

where F contains no power of t beyond t^{e-1}. Thus, as before, we find that

$$F(At-B) = t^e R(t).$$

Hence, if

$$\begin{aligned} FB &= y_1+y_2t+\ldots+y_et^{e-1}, \\ FA &= FB(\xi) \\ &= \eta_1+\eta_2t+\ldots+\eta_et^{e-1}, \end{aligned}$$

we find the equations

$$0 = y_1, \quad \eta_1 = y_2, \quad \ldots, \quad \eta_{r-1} = y_r \quad (r = 2, 3, \ldots, e_k).$$

It will be observed that in the reduction of the substitution just investigated we have a number of arbitrary constants at our disposal. But it is known that in the reduction of a substitution, the roots of the characteristic equation not being repeated, there is only one arbitrary constant in the value of each y; by the investigation above we have, apparently, n. This disparity is accounted for by a theorem of Stickelberger's;* which is to the effect that the bilinear form

$$F_{k-1} = \begin{vmatrix} d_{11}, & \ldots, & d_{1n}, & p_1', & \ldots, & p_1^{k-1}, & x_1 \\ \ldots & \ldots & \ldots & & \ldots & & \ldots \\ d_{n1}, & \ldots, & d_{nn}, & p_n', & \ldots, & p_n^{k-1}, & x_n \\ q_1', & \ldots, & q_n', & 0, & \ldots, & 0, & 0 \\ \ldots & \ldots & \ldots & & \ldots & & \\ q_1^{k-1}, & \ldots, & q_n^{k-1}, & 0, & \ldots, & 0, & 0 \\ y_1, & \ldots, & y_n, & 0, & \ldots, & 0, & 0 \end{vmatrix},$$

$$d_{rs} = a_{rs}-(c+t)\,b_{rs},$$

when expanded in powers of t has for the coefficient of the lowest power of t in the expansion a bilinear form of rank l, where l is determined by

$$e_k = e_{k+1} = \ldots = e_{k+l-1} > e_{k+l}.$$

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* *Crelle* (1879), Bd. LXXXVI., p. 42, Satz. vii.; Muth's Elementartheiler, pp. 135–139.

Consequently in the case of non-repeated roots the form

$$\begin{vmatrix} d_{11}, & \ldots, & d_{1n}, & b_{11}x_1+\ldots+b_{1n}x_n \\ \ldots & \ldots & \ldots & \ldots \\ d_{n1}, & \ldots, & d_{nn}, & b_{n1}x_1+\ldots+b_{nn}x_n \\ w_1, & \ldots, & w_n, & 0 \end{vmatrix},$$

$$d_{rs} = a_{rs} - \lambda b_{rs},$$

will contain only one effective arbitrary constant; for here

$$e_1 = 1 > e_2,$$

and thus Stickelberger's theorem shows that the rank is unity.

The following proof* and generalization of Stickelberger's theorem are new, so far as I am aware. Following Darboux, Stickelberger has proved that we have an equation

$$\frac{F_{k-1}}{\Delta_{k-1}} - \frac{F_k}{\Delta_k} = \frac{X_k Y_k}{\Delta_{k-1}\Delta_k},$$

where F_{k-1} is the bilinear form as defined above; and X_k (Y_k) is found from F_{k-1} by replacing the y's (x's) with $q_1^k, \ldots, q_n^k$ ($p_1^k, \ldots, p_n^k$).

It follows that

$$\frac{F_{k-1}}{\Delta_{k-1}} = \frac{X_k Y_k}{\Delta_{k-1}\Delta_k} + \ldots + \frac{X_n Y_n}{\Delta_{n-1}\Delta_n}.$$

Now, from the definitions, we shall have

$$X_r/\Delta_r = L_0 + L_1 t + L_2 t^2 + \ldots$$

and $$Y_r/\Delta_r = M_0 + M_1 t + M_2 t^2 + \ldots,$$

$$\Delta_{r-1}/\Delta_r = t^e(\alpha_0 + \alpha_1 t + \ldots) \quad (e = e_r).$$

By hypothesis, $$e_k = e_{k+1} = \ldots = e_{k+l-1} > e_{k+l},$$

and hence, on expanding the value of F_{k-1}/Δ_{k-1}, the coefficient of t^{-e} is a sum of l products of the type $L_0 M_0$; or the rank of the coefficient is l. Here the products are ordinary, not symbolical.

The extension which I make is found by considering the coefficient

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[*] Stickelberger proves only a particular case, referring to a paper by Frobenius; the reference is to a proof of Sylvester's theorem (*Phil. Mag.*, 1851) on *minors of any order.* Muth gives a proof (*l.c.*) which is attributed to Frobenius. *I have not met with any other proof in the papers I have consulted.*

of t^{-1} in the expansion; this contains e_k products from the fraction $(X_kY_k)/(\Delta_{k-1}\Delta_k)$, and, in general, e_r from $(X_rY_r)/(\Delta_{r-1}\Delta_r)$. So, in all, this coefficient contains products in number

$$e_k+e_{k+1}+\ldots = (h_{k-1}-h_k)+(h_k-h_{k+1})+\ldots = h_{k-1},$$

and thus h_{k-1} is the rank of the coefficient of t^{-1} in $(X_kY_k)/(\Delta_{k-1}\Delta_k)$; and this will be the highest rank of all the first e_k terms in the expansion of F_{k-1}/Δ_{k-1}.*

This fact will be of service in counting the number of effective arbitrary constants in the reduced forms.

Suppose, in the next place, that A, B are so related that

$$| A-\lambda B | \equiv 0$$

for all values of λ; so that $| A | = 0 = | B |$. Then we must examine the first, second, third, ... minors of $| A-\lambda B |$; let us suppose that all the $(k-1)^{\text{th}}$ minors are identically zero, but not all the k^{th}; so that at least one k^{th} minor is not zero identically.

Then, if we write

$$\Delta_r = \begin{vmatrix} d_{11}, \ldots, d_{1n}, & p_1', \ldots, p_1^r \\ \ldots \quad\quad \ldots & \ldots \\ d_{n1}, \ldots, d_{nn}, & p_n', \ldots, p_n^r \\ q_1', \ldots, q_n', & 0, \ldots, 0 \\ \ldots \quad\quad \ldots & \ldots \\ q_1^r, \ldots, q_n^r, & 0, \ldots, 0 \end{vmatrix},$$

where $$d_{rs} = a_{rs}-\lambda b_{rs},$$

we shall have $$\Delta_0 = 0 = \Delta_1 = \ldots = \Delta_{k-1};$$

but $$\Delta_k \neq 0.$$

Now it is well known, by the theory of zero determinants, that we can find k sets of quantities (functions of the d's), such that for each set

$$f_1d_{1r}+f_2d_{2r}+\ldots+f_nd_{nr} = 0 \quad (r = 1, 2, \ldots, n),$$

and these k sets will be independent of each other. We shall suppose the k sets so determined that, if $p_1, \ldots, p_n$ be any quantities

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* *In this, h_r denotes the index* of the highest power of t which divides Δ_r.

(not regarded as functions of λ), then the k expressions

$$p_1 f_1^r + \dots + p_n f_n^r \text{*} \quad (r = 1, 2, \dots, k)$$

are to be each of the lowest possible degree in λ. It is then known that these k sets of f's are linearly independent.

In like manner we determine k sets of y's, such that for each set

$$g_1 d_{r1} + g_2 d_{r2} + \dots + g_n d_{rn} = 0 \quad (r = 1, 2, \dots, n),$$

and the k expressions

$$q_1 g_1^r + \dots + q_n g_n^r \qquad (r = 1, 2, \dots, k)$$

are each of the lowest possible degree in λ.

Now multiply the first n rows of Δ_k by $f_1^{\prime}, \dots, f_n^{\prime}$, add the results, and replace the first row of Δ_k by the sums so found. In like manner, replace the second row by similar quantities obtained by using $f_1^2, \dots, f_n^2$; and so on to the kth row. The determinant so formed will be $(f_1^1 f_2^2 \dots f_k^k)$ times Δ_k.

Similarly, modify the first k columns by using the k sets of g's. We finally have $\Delta_k (f_1^{\prime} f_2^2 \dots f_k^k)(g_1^{\prime} \dots g_k^k)$ equal to a determinant of the form

$$\left|\begin{array}{cccccc}
0, \dots, 0, & & 0, \dots, 0, & & P_1^{\prime}, \dots, P_1^k & \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0, \dots, 0, & & 0, \dots, 0, & & P_k^{\prime}, \dots, P_k^k & \\
0, \dots, 0, & & d_{k+1, k+1}, \dots, d_{k+1, n}, & & p_{k+1}^{\prime}, \dots, p_{k+1}^k & \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0, \dots, 0, & & d_{n, k+1}, \dots, d_{nn}, & & p_n^{\prime}, \dots, p_k & \\
Q_1^{\prime}, \dots, Q_k^{\prime}, & & q_{k+1}^{\prime}, \dots, q_n^{\prime}, & & 0, \dots, 0 & \\
\dots & \dots & \dots & \dots & \dots & \dots \\
Q_1^k, \dots, Q_k^k, & & q_{k+1}^k, \dots, q_n^k, & & 0, \dots, 0 &
\end{array}\right|,$$

in which we have written

$$\left.\begin{array}{l} P_s^r = p_1^r f_1^s + \dots + p_n^r f_n^s \\ Q_s^r = q_1^r f_1^s + \dots + q_n^r f_n^s \end{array}\right\} \quad (r, s = 1, 2, \dots, k).$$

It follows that $\qquad \Delta_k = E_1 E_2 E_3,$

---

* *Here the indices r applied to the f's will distinguish between the sets of f's; they do not refer to powers.*

where $E_1 = | P |$, a determinant of k rows and columns, $E_2 = | Q |$, a determinant of the same form, and E_3 is the quotient of a certain k^{th} minor of $(A-\lambda B)$ by the product $(f_1' g_1' \dots f_k^k g_k^k)$. Now E_1, E_2 will not be divisible by any function of λ which does not contain p's or q's; and the denominator of E_3 will be a function only of λ. Hence, as Δ_k is an integral function of λ, so also is E_3.*

On examining the value of P_s^r it is clear that its degree in λ depends only on s (not on r), and let this degree be denoted by a_s; similarly, let the degree of Q_s^r be denoted by β_s. Then the numbers $a_1, \dots, a_k, \beta_1, \dots, \beta_k$ will be the two sets of *Minimalgradzahlen* belonging to $| A-\lambda B |$; and Kronecker's invariants will be

$$2a_1+1, \ \dots, \ 2a_k+1, \ 2\beta_1+1, \ \dots, \ 2\beta_k+1.$$

Let us now suppose a particular set of values of the arbitrary p's are chosen; and to do this let us assume

$$P_r^r \equiv 1, \quad P_s^r \equiv 0 \quad (s = 1, 2, \dots, r-1, r+1, \dots, k)$$

for all possible values of r, *i.e.*, from 1 to k. The number of conditions to determine each set of p's will be

$$(a_1+1)+(a_2+1)+\dots+(a_k+1),$$

for in P_s^r there are (a_s+1) coefficients of the various powers of λ.

Now $(a_1+a_2+\dots+a_k)$ is the index of the highest power of λ in E_1, and is thus not greater than $(n-k)$, which is the highest power of λ in Δ_k. Thus

$$(a_1+1)+\dots+(a_k+1) \overset{=}{<} n;$$

and n is the number of p's in each set, or we have at least enough p's to satisfy all the conditions imposed on the P's.

Now in E_1 replace $p_1', \dots, p_n'$ by the quantities $(b_{11}x_1+\dots+b_{1n}x_n), \dots, (b_{n1}x_1+\dots+b_{nn}x_n)$ respectively; the other p's retaining the values as just determined. The value of E_1 will then be

$$\begin{vmatrix} F_1, & 0, & 0, & \dots, & 0 \\ F_2, & 1, & 0, & \dots, & 0 \\ F_3, & 0, & 1, & \dots, & 0 \\ \dots & & \dots & & \dots \\ F_k, & 0, & 0, & \dots, & 1 \end{vmatrix},$$

---

* It may be useful to point out that it is not necessary (as a rule) to calculate all the f's and g's; Δ_k will generally arrange itself in the proper form when we have to reduce a given numerical substitution.

where $$F_r = f_1^r\,(b_{11}x_1 + \dots + b_{1n}x_n) + \dots + f_n^r\,(b_{n1}x_1 + \dots + b_{nn}x_n),$$

and consequently E_1 will simply reduce to F_1. The degree of F_1 in λ will be the degree of

$$f_1'\,(b_{11}x_1 + \dots + b_{1n}x_n) + \dots + f_n'\,(b_{n1}x_1 + \dots + b_{nn}x_n),$$

which is a_1. Consequently, if we write

$$F_1 = y_1 + y_2\lambda + y_3\lambda^2 + \dots,$$

we shall obtain (a_1+1) linear functions of the x's.

But, returning to the original definition of E_1 (by the aid of Δ_k), we have that the product $F_1E_2E_3$ is equal to Δ_k with the column $p_1', \dots, p_n'$ replaced by $(b_{11}x_1 + \dots + b_{1n}x_n)$, ..., $(b_{n1}x_1 + \dots + b_{nn}x_n)$. Similarly, we express $\Phi_1E_2E_3$, where Φ_1 is the value of F_1 with the ξ's replacing the x's.

Hence $(\Phi_1 - \lambda F_1)\,E_2E_3$ is equal to a determinant of the type Δ_k, but with p_r' replaced by

$$\begin{aligned}(b_{r1}\xi_1 + \dots + b_{rn}\xi_n) - \lambda\,(b_{r1}x_1 + \dots + b_{rn}x_n)\\ &= (a_{r1}x_1 + \dots + a_{rn}x_n) - \lambda\,(b_{r1}x_1 + \dots + b_{rn}x_n)\\ &= d_{r1}x_1 + \dots + d_{rn}x_n,\end{aligned}$$

in reducing which we have used the original linear equations; and here, as in Δ_k, $d_{rs} = a_{rs} - \lambda b_{rs}$. We now multiply the first, ..., n^{th} columns of the last determinant by $x_1, \dots, x_n$ and subtract from the $(n+1)^{\text{th}}$ column. This column then contains n zeroes followed by k terms of the type $-(q_1^rx_1 + \dots + q_n^rx_n)$. We now expand this determinant in terms of the elements of the $(n+1)^{\text{th}}$ column; and it will be seen that the co-factor of each non-zero element is a determinant of the type Δ_{k-1}, which will by hypothesis vanish identically.

Hence $$\Phi_1 - \lambda F_1 \equiv 0,$$

or, if $$\Phi_1 = \eta_1 + \eta_2\lambda + \dots,$$

we have the (a_1+2) equations

$$\eta_1 = 0, \quad \eta_2 = y_1, \quad \dots, \quad \eta_{m+1} = y_m, \quad 0 = y_{m+1}$$

where $$m = a_1.$$

In the same way, if the quantities $p_1^r, \dots, p_n^r$ in E_1 be replaced by $(b_{11}x_1 + \dots + b_{1n}x_n)$, ..., $(b_{n1}x_1 + \dots + b_{nn}x_n)$, we shall obtain (a_r+1) y's and (a_r+1) η's which must satisfy (a_r+2) equations of the type just *written.*

In all we get $(a_1+a_2+\ldots+a_k+k)$ y's and $(a_1+a_2+\ldots+a_k+2k)$ equations by treating all the various sets of p's in this way.

We pass to the consideration of corresponding results due to E_2; in the first place, let the q's be determined in such a way that

$$Q_r^r \equiv 1, \quad Q_s^r \equiv 0 \quad (s = 1, 2, \ldots, r-1, r+1, \ldots, k).$$

We shall, however, require further sets of variables typified by γ_r^s, and the definition of these will be that

$$\Gamma_r^r \equiv \lambda^{\beta_r}, \quad \Gamma_s^r \equiv 0 \quad (s = 1, 2, \ldots, r-1, r+1, \ldots, k),$$

where Γ is formed from the γ's just as Q is from the q's. The argument applied above in determining the number of conditions to be satisfied by the p's will show that we have sufficient q's and γ's to satisfy all these conditions.

If, now, we take Δ_k and replace one row of q's (say the r^{th}) by a row of γ's (say by $\gamma_1', \ldots, \gamma_n'$), the result will be (E_1E_2) times E_2, the corresponding substitution having been made in E_2. But (unless $r = 1$) the value of E_2 will then be zero; for the first and r^{th} rows will each have zeroes in all the spaces except the first. If $r = 1$, the value of E_2 will be λ^{β_1}. Thus, in general, if the r^{th} row of q's in Δ_k be replaced by $\gamma_1', \ldots, \gamma_n'$, the result will be zero; but, if the first row be similarly replaced, the result is $(E_1E_2)\,\lambda^{\beta_1}$.

Now consider the value of

$$Y = \begin{vmatrix} c_{11}, \ldots, c_{1n}, & p_1', \ldots, p_1^k, & (b_{11}x_1+\ldots+b_{1n}x_n) \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ c_{n1}, \ldots, c_{nn}, & p_n', \ldots, p_n^k, & (b_{n1}x_1+\ldots+b_{nn}x_n) \\ q_1', \ldots, q_n', & 0, \ldots, 0, & 0 \\ \cdots \quad \cdots & \cdots \quad \cdots & \cdots \quad \cdots \\ q_1^k, \ldots, q_n^k, & 0, \ldots, 0, & 0 \\ \gamma_1', \ldots, \gamma_n', & 0, \ldots, 0, & 0 \end{vmatrix},$$

and let us similarly form H from the ξ's. Then, taking $(H-\lambda Y)$ and treating it in a similar way to the determinants before considered, we reduce the last column to n zeroes, followed by the $(k+1)$ quantities

$$-(q_1'x_1+\ldots+q_n'x_1), \quad \ldots, \quad -(q_1^kx_1+\ldots+q_n^kx_n), \quad -(\gamma_1'x_1+\ldots+\gamma_n'x_n).$$

Now, expanding the determinant, we see that the co-factor of the last element is Δ_k; while the co-factor of the element $-(q_1^rx_1+\ldots+q_n^rx_n)$ will be $(-\Delta_k)$ with the quantities $(q_1^r, \ldots, q_n^r)$ replaced by $(\gamma_1', \ldots, \gamma_n')$; *and these co-factors will* accordingly vanish, all but the first.

Hence we see that

$$H-\lambda Y = E_1 E_3 \left[\lambda^{\beta_1}(q_1' x_1+\ldots+q_n' x_n)-(\gamma_1' x_1+\ldots+\gamma_n' x_n)\right].$$

Now the p's were determined so as to give $E_1 = 1$, and thus, if we write*

$$Y/E_3 = y_1+y_2\lambda+y_3\lambda^2+\ldots,$$
$$H/E_3 = \eta_1+\eta_2\lambda+\eta_3\lambda^2+\ldots,$$

we see that the typical equations between the η's and y's are

$$\eta_2 = y_1, \quad \eta_r = y_{r-1} \quad (r = 3, \ldots, \beta_1)$$

We have thus (β_1-1) equations between β_1 y's and β_1 η's.

Similarly, we derive y's and η's from each set of γ's. In all we obtain $(\beta_1+\beta_2+\ldots+\beta_k)-k$ equations between $(\beta_1+\beta_2+\ldots+\beta_k)$ y's and the same number of η's.

Combining with the numbers found by using E_1, we see that from E_1 and E_2 we have obtained altogether $(\Sigma\alpha+\Sigma\beta+k)$ equations connecting the same number of y's and of η's. But, if ϵ be the highest index of λ in E_3, then

$$\Sigma\alpha+\Sigma\beta+\epsilon = n-k,$$

for this is the degree of Δ_k in λ. Hence we have already obtained $(n-\epsilon)$ equations amongst $(n-\epsilon)$ y's and the same number of η's.

Passing to E_3, we can obtain exactly ϵ more equations by the method originally set forth at the beginning of this note; that is, we treat E_3 as if it had been the original Δ_0, and thus derive ϵ more equations.

In all, then, we have n reduced equations between n linear functions of the x's and the same functions of the ξ's.

2. *Dynamical Application.*

The problem of integrating a system of linear differential equations with constant coefficients is at once reduced to its simplest form by the results of our investigations. As the theory has been considered by many writers, it will be sufficient to refer to Herr Muth's *Elementartheiler*, § 16, where a list of papers on this subject will be found; amongst others, see Weierstrass, *Ges. Werke*, Bd. II., pp. 75–76.

I wish, however, to examine in more detail a special problem of dynamics. This is the problem of small oscillations about a state of steady motion, treated by the aid of Hamilton's canonical equations of motion. The suggestion of using Hamilton's (instead of Lagrange's) equations is due to Mr. E. T. Whittaker, who proposed to me the problem of reducing (in this case) the Hamiltonian function to a canonical form by a contact-transformation. The solution of this problem I hope to publish as an appendix to Mr. Whittaker's paper.

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* *If λ is a factor of E_3*, there will be, in general, terms containing negative *powers of λ in Y/E_3*; *such* terms will not, however, affect the results.

In a system of n degrees of freedom, let us denote by $x_1, \ldots, x_n$ the n coordinates of the system chosen so as to be zero in the state of steady motion; and let $\xi_1, \ldots, \xi_n$ be the corresponding momenta. Hamilton's equations are then typified by

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}, \quad -\frac{d\xi}{dt} = \frac{\partial H}{\partial x} \quad \text{(for all suffixes)},$$

where H is the sum of the quadratic terms in the Hamiltonian function.*

It will be seen at once that the form of solution is determined by the invariant-factors of the determinant $|\, K-\lambda A \,|$, where K is the symmetrical bilinear form, in $2n$ pairs of variables,

$$K = \Sigma y_r \frac{\partial H}{\partial x_r} + \Sigma \eta_r \frac{\partial H}{\partial \xi_r},$$

and A is the alternate bilinear form

$$A = \Sigma\, (x_r \eta_r - y_r \xi_r).$$

We observe that $|\, A \,| = 1$, and thus we have no exceptional cases of $|\, K-\lambda A \,| \equiv 0$.

In order that the solution of Hamilton's equations may contain no powers of t, it is necessary and sufficient that the invariant-factors of $|\, K-\lambda A \,|$ should be *linear;* and, in order that the solution may depend only on harmonic functions of t, the roots of $|\, K-\lambda A \,| = 0$ must be pure imaginaries, and not complex.

I now proceed to show that, if H be never negative for any real values of the x's and ξ's, then both these conditions are satisfied. According to this hypothesis, K is a positive definite form, and K can be put in the form $L'L$, where L is any one of an infinity of real bilinear forms and L' is the conjugate of L [obtained from L by changing each x (or y) to the corresponding y (or x) and each ξ (or η) to η (or ξ)]. This is only another way of stating the fact that H can be reduced to a sum of *real* squares in an infinity of ways.

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* Aug. 11, 1900.—I find that this system of equations has been considered by Weierstrass (*Berliner Monatsberichte*, 1879, p. 430), who obtains results similar to mine. This paper has not yet appeared in Weierstrass's *Collected Works*, which I consulted during my investigations. Weierstrass gives results for alternate forms found independently by Mr. Taber (see footnote, p. 95).

We shall first prove the following :—

Lemma.—If K be a positive definite real form and F any bilinear form with complex coefficients, then, if

$$F'KF_1 = 0,$$

where F_1 is obtained from F by changing $+i$ to $-i$,

we must have $$KF = 0 \quad\text{and}\quad F'K = 0.$$

For let $$F = P + iQ;$$

then $$F_1 = P - iQ \quad\text{and}\quad F' = P' + iQ'.$$

Hence $$(P' + iQ')\,K\,(P - iQ) = 0$$

or $$P'KP + Q'KQ + i\,(Q'KP - P'KQ) = 0.$$

But, as just explained, $K = L'L$, and thus

$$P'KP = P'L'LP = (LP)'(LP).$$

This form is always positive, unless $LP = 0$; similarly, $Q'KQ$ is positive, unless $LQ = 0$. Hence $F'KF_1$ can only vanish if $LP = 0$ and $LQ = 0$, or if $LF = 0$; and this gives $KF = 0$ and also $KF_1 = 0$.

We have further the following theorem (given by Frobenius and Stickelberger) :—If A, B are any two bilinear forms such that $|\,A - \lambda B\,| \not\equiv 0$, and if the form $(A - \lambda B)^{-1}$ be expanded in an ascending series of powers of $(\lambda - c)$, the first term in the expansion is of the form $F/(\lambda - c)^e$, where $(\lambda - c)^e$ is the first of a series of invariant-factors of the determinant $|\,A - \lambda B\,|$. (See p. 81, footnote.)

We apply these theorems to deduce properties of the invariant-factors of $|\,K - \lambda A\,|$, where K is a positive definite form, and A is an alternate form, such that $|\,K - \lambda A\,| \not\equiv 0$.

If, then, $(\lambda - c)^e$ be the first invariant-factor of $|\,K - \lambda A\,|$, we have

$$(K - \lambda A)^{-1} = \frac{F}{(\lambda - c)^e} + \frac{G}{(\lambda - c)^{e-1}} + \ldots$$

or
$$\begin{aligned}(\lambda - c)^e E &= F(K - \lambda A) + (\lambda - c)\,G(K - \lambda A) + \ldots\\ &= (K - \lambda A)\,F + (\lambda - c)\,(K - \lambda A)\,G + \ldots,\end{aligned}$$

where E is the unit-form.* Now, by the properties of invariant-factors, $e \overset{=}{>} 1$; and thus

$$0 = F(K - cA) = (K - cA)\,F.$$

---

* *Einheitsform;* = here $\Sigma\,(x_r y_r + \xi_r \eta_r)$.

If c be complex, and c_1 be obtained from c by changing $+i$ to $-i$, then $(\lambda - c_1)$ is also a factor of $|K-\lambda A|$, and hence we have

$$0 = F_1(K-c_1A) = (K-c_1A)F_1$$

where F_1 is obtained from F by changing $+i$ to $-i$. But, taking the forms conjugate to the first pair of equations, we have

$$0 = (K'-cA')F' = F'(K'-cA');$$

but $$K' = K \quad \text{and} \quad A' = -A,$$

for K is symmetrical, but A is alternate.

Hence $$F'KF_1 = c_1F'AF_1 = \text{also } -cF'AF_1,$$

and thus *either* $$c+c_1 = 0 \quad \textit{or} \quad F'KF_1 = 0.$$

If $$F'KF_1 = 0,$$

we have $$KF = 0,$$

by the lemma proved above (p. 94); hence

$$AF = 0,$$

and thus $$(K-\lambda A)F = 0.$$

Now $$|K-\lambda A| \not\equiv 0;$$

thus, if $$(K-\lambda A)F = 0,$$

$$F = 0,$$

which is impossible. Hence we have

$$c+c_1 = 0,$$

and the real part of c is zero; or the roots of

$$|K-\lambda A| = 0$$

are pure imaginaries.*

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* This seems to have been noticed for the first time in England by Mr. Henry Taber [*Proc. Lond. Math. Soc.* (1891), Vol. xxii., pp. 449–469], who has also given a proof of the property that the invariant-factors of $|K-\lambda A|$ are linear (stated in the equivalent form "nullity = vacuity"). I had not become acquainted with Mr. Taber's work when this paper was originally written; I ought to remark that Prof. Elliott had arrived at the former result (also independently) (*Quar. Jour. of Math.*, Dec., 1899). In this connexion I may point out that in the above investigation, if A be assumed symmetrical, we get $c = c_1$, or c is real. This seems to be the shortest arrangement of Prof. Elliott's proof of the reality of all the roots of $|K-\lambda A|$; similar methods have been used by Cauchy, Thomson and Tait, Frobenius, Heaviside, and Taber. Mr. Taber's results for symmetrical and orthogonal forms had been anticipated by Weierstrass [*Berliner Monatsberichte* (1858), p. 215 = *Ges. Werke*, Bd. i., p. 241] and Frobenius [*Crelle* (1878), Bd. lxxxiv., p. 53]. See also footnote, p. 93.

We have thus that

$$(K-\lambda A)^{-1} = \frac{F}{(\lambda-c)^e} + \frac{F_1}{(\lambda+c)^e} + \dots,$$

and, taking the form conjugate to each side, we obtain, since

$$K' = K \quad \text{and} \quad A' = -A,$$

$$(K+\lambda A)^{-1} = \frac{F'}{(\lambda-c)^e} + \frac{F_1'}{(\lambda+c)^e} + \dots.$$

Changing the sign of λ, we have

$$(K-\lambda A)^{-1} = \frac{F'}{(-\lambda-c)^e} + \frac{F_1'}{(-\lambda+c)^e} + \dots.$$

Thus, on comparison of the two expansions for $(K-\lambda A)^{-1}$, we have

$$F = (-1)^e F_1',$$

$$F' = (-1)^e F_1.$$

But we have

$$(\lambda-c)^e E = F(K-\lambda A) + (\lambda-c)\, G(K-\lambda A) + \dots$$

$$= (K-\lambda A)\, F + (\lambda-c)(K-\lambda A)\, G + \dots,$$

and thus, if $e > 1$, we find

$$F(K-cA) = 0 = (K-cA)\, F$$

and $$-FA + G(K-cA) = 0 = -AF + (K-cA)\, G.$$

From these we have

$$FAF = F(K-cA)G = 0,$$

and thus $$FKF = cFAF = 0.$$

But $$FKF = (-1)^e FKF_1',$$

and, as we have proved on p. 94, this will only vanish if

$$FK = 0,$$

or if $$cFA = 0.$$

If, then, $c \neq 0$, the assumption $e > 1$ leads to

$$FA = 0,$$

and so to $$F(K-\lambda A) = 0 \quad \text{and} \quad |K-\lambda A| \neq 0;$$

so this equation is untenable. Thus $e = 1$, unless $c = 0$.

We shall now show that if $c = 0$, we may have

$$e = 2,$$

but that $e>2$ is untenable. For, if $c=0$ and $e>2$, we have

$$FK = 0 = KF,$$

$$-FA+GK = 0 = -AF+KG,$$

$$-GA+HK = 0 = -AG+KH.\text{*}$$

These give $$GKG = FAG = FKH = 0,$$

and, as in the previous work, we show that

$$G = (-1)^{e-1} G_1';$$

so we are to have $$G_1' KG = 0,$$

which involves $$KG = 0 \quad \text{or} \quad AF = 0,$$

and thus $$(K-\lambda A)\, F = 0,$$

which is impossible, since $| K-\lambda A | \not\equiv 0$.

Thus $| K-\lambda A |$ may have a first invariant-factor of the form λ^2; but of no higher power of λ.

Now it is well known that every first invariant-factor is divisible by every other invariant-factor which belongs to the same root; and hence no invariant-factor of $| K-\lambda A |$ can be other than linear, except those which belong to a zero root of $| K-\lambda A | = 0$, and these may be of the form λ^2.

Finally, we conclude, on returning to the dynamical problem, that the expressions for all the coordinates will contain only simple harmonic functions of the time, provided that H (the quadratic terms in the Hamiltonian function) is always positive and that $| H | \not\equiv 0$; if $| H | = 0$, then the first power of the time may appear alone (not multiplied by sines or cosines).

This result agrees with Weierstrass's (see footnote on p. 93), and it is verified in the special case when H contains no terms of the type $x_r \xi_s$. This special case corresponds to small oscillations about a position of equilibrium; in this case it has been shown by Weierstrass† that the presence of equal periods does not introduce powers of t, provided that either $H(x, 0)$ or $H(0, \xi)$‡ is always positive (with the same exception as above in case $|H| = 0$); and this is always satisfied in real dynamical problems.

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* H is not here the Hamiltonian function, but a bilinear form arising from the expansion of $(K-\lambda A)^{-1}$ in powers of λ.

† *Berliner Monatsberichte*, 1858, p. 215; 1868, p. 336. *Gesammelte Werke*, Bd. I., p. 241, and Bd. II., p. 42.

‡ I use $H(x, 0)$ *to denote the* value of H when all the ξ's are made zero.

Weierstrass's condition indicates that the condition that H should be always positive is sufficient, but not necessary, to ensure the absence of powers of t in the case of equal periods, and it is easy to show by the examination of a few special problems that this condition is sufficient, but not necessary, to ensure the reality of the periods. It will be necessary to examine cases in which terms of the type $x_r \xi_s$ do appear; for otherwise our condition simply reduces to the condition that the potential energy should be positive, which is well known to be the necessary and sufficient condition for real periods.

Take, for example, the case of a top spinning nearly vertically; if $(l, m, 1)$ are the cosines of the axis, the value of Routh's modified Lagrangian function is of the type*

$$L = \tfrac{1}{2}\left[A(\dot{l}^2+\dot{m}^2)+2B(l\dot{m}-\dot{l}m)+C(l^2+m^2)\right],$$

where l, m are of course small.

The condition for stability is readily found to be

$$B^2 > AC,$$

by forming the equations of motion in the ordinary way. But

$$2H = 2(p\dot{l}+q\dot{m})-L,$$

where
$$p = \frac{\partial L}{\partial \dot{l}}, \quad q = \frac{\partial L}{\partial \dot{m}},$$

and thus
$$2H = \frac{1}{A}(Bm+p)^2+\frac{1}{A}(Bl-q)^2-C(l^2+m^2).$$

For this to be always positive we must have C negative and A positive, which is sufficient, but not necessary, to ensure

$$B^2 > AC,$$

or to ensure stability.

3. *The Canonical Reduction of a Pair of Bilinear Forms.*

The problem here considered has been solved in various forms by numerous writers; some of the best known works on the subject are—

Weierstrass, *Berliner Monatsberichte*, 1858, p. 207, and 1868, p. 310; *Gesammelte Werke*, Bd. I., p. 233, and Bd. II., p. 19.

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* A, B, C are here constants of inertia, not bilinear forms.

Jordan, *Liouville's Journal*, 1874, t. xix. (2me Série), pp. 35 and 397.
Darboux, *ibid.*, p. 347.
Stickelberger, *Crelle's Journal*, 1879, Bd. lxxxvi., p. 20.
Frobenius, *ibid.*, p. 202.
Kronecker, *Berliner Monatsberichte*, 1868, p. 339; 1874, pp. 59, 149, 206 = *Gesammelte Werke*, Bd. i., pp. 163, 349. *Berliner Sitzungsberichte*, 1890, pp. 1225, 1375; 1891, pp. 9, 33.
Sauvage, *Annales de l'Ecole Normale Supérieure*, 1891 and 1893.
Landsberg, *Crelle's Journal*, 1896, Bd. cxvi., p. 331.
Muth, *Theorie der Elementartheiler*, Leipzig, 1899, §§ 6–8.

Kronecker's and Jordan's methods are in every case based on a step-by-step process, which is doubtless very convenient for many problems, but is nevertheless subject to some disadvantages in case it is desired to put down the complete substitutions.

Frobenius's and Landsberg's methods differ from the rest; but all the other authors that I have been able to consult have followed Weierstrass in determining the variables of the reduced forms (typified by ξ, η) as linear functions of the variables of the given forms (typified by x, y). In what follows I have expressed (x, y) as linear functions of (ξ, η). That is, I have found the substitutions reciprocal to those given by the writers alluded to. The theoretical investigation of these substitutions is longer than Weierstrass's; but, on the other hand, the actual calculation of the coefficients would be somewhat easier in practical cases. In fact my coefficients are found by expanding the quotient of two polynomials in powers of a parameter t; the substitutions formerly given are obtained by expanding, in like manner, the quotient of one polynomial by the square root of the product of two others; and, if the expansion has to be carried out to several terms, the difference in the labour of calculation may be considerable. There is, however, the disadvantage that in my solution certain arbitrary constants have to be chosen so as to make certain forms zero.

Another point of some advantage is that, if

$$|A| = 0 \quad \text{or} \quad |B| = 0,$$

the requisite changes in the method are fewer than in the Weierstrassian form.

It may be well to point out here that, if the symbolical product of three bilinear forms PAQ be taken in Frobenius's manner (or of three matrices in Cayley's), the result is equivalent to making two linear

$$x = \frac{\partial Q}{\partial y} = \text{a definite constant}$$

d $$y = \frac{\partial Q}{\partial x} = \text{another definite constant.}$$

We begin with the simple case of non-repeated root

$$\Delta_0 = | A-\lambda B | = 0;$$

; the n roots be $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let R be the a
$-\lambda B$), or let R be $\Sigma a_{rs} x_r y_s'$, where a_{rs} is the minor
$A-\lambda B$ | . Then we know that

$$R(A-\lambda B) = (A-\lambda B) R = \Delta_0 E,$$

here the products are to be formed according to F
lical method of multiplication,* and E is the unit
rm) $\Sigma x_r y_r$. Suppose that $R_1, \ldots, R_n$ are the values
placed by $\lambda_1, \ldots, \lambda_n$; then

$$R_1 (A-\lambda_1 B) = (A-\lambda_1 B) R_1 = 0,$$

&c.

ence $$R_1 (A-\lambda B) = (\lambda_1-\lambda) R_1 B,$$

$$(A-\lambda B) R_1 = (\lambda_1-\lambda) BR_1,$$

But the product $R_1(A-\lambda B)R_2$ must be linear in λ; and it has $(\lambda_1-\lambda)$, $(\lambda_2-\lambda)$ as factors. Thus

$$R_1(A-\lambda B)R_2 = 0.$$

Similarly $$R_r(A-\lambda B)R_s = 0 \quad (r \neq s)$$

and $$R_r(A-\lambda B)R_r = (\lambda_r-\lambda)R_rBR_r.$$

In a symbolical product such as $R_r(A-\lambda B)R_s$, the final variables left in the complete product will be the x's of R_r and the y's of R_s; if, then, these are replaced by constants, the resulting product will be constant. Let us write S_r, (T_r) for the value of R_r when the x's (y's) are replaced by p's (q's). Thus write, for example,

$$S_1 = \theta_1 \Sigma a_{rs} p_r y_s \quad (\lambda = \lambda_1),$$

$$T_1 = \phi_1 \Sigma a_{rs} x_r q_s \quad (\lambda = \lambda_1),$$

where θ, ϕ are constants.

Then we have $$S_r(A-\lambda B)T_s = 0 \quad (r \neq s)$$

and $$S_r(A-\lambda B)T_r = \lambda_r-\lambda,$$

provided that the value of the product $\theta_r\phi_r$ is adjusted so as to give

$$S_rBT_r = 1.$$

If, now, we write $$P = \Sigma x_r S_r, \quad Q = \Sigma y_r T_r,$$

we shall have $$P(A-\lambda B)Q = \Sigma x_r y_s S_r(A-\lambda B)T_s$$

$$= \Sigma(\lambda_r-\lambda)x_r y_r,$$

by virtue of the relations found above.

We have, then, simultaneously reduced A, B by the substitutions P, Q, so that

$$PAQ = \Sigma \lambda_r x_r y_r,$$

$$PBQ = \Sigma x_r y_r,$$

which are the canonical forms for the simple case considered.

This method (or equivalent ones in other notations) has been used by many writers. It is only reproduced here by way of introduction to what follows.

Let us now suppose that the roots of

$$\Delta_0 = |A-\lambda B| = 0$$

are not all distinct; and let us take $\lambda = c$ to be a root which appears h_0 *times in* $\Delta_0 = 0$, *and* h_1 *times* in *every* first minor of Δ_0.

$$R(t)\left[A-(c+t)B\right]=\Delta_0(t)\,E.$$

$$\left[A-(c+v)B\right]R(v)=\Delta_0(v)\,E.$$

$$R(t)\left[A-(c+t)B\right]R(v)=\Delta_0(t)\,R(v).$$

$$R(t)\left[A-(c+v)B\right]R(v)=\Delta_0(v)\,R(t).$$

efore, we may replace the x's in $R(t)$ by p's and the
s ; let the forms so found be denoted by $S(t)$, $T($
the former notation ; further, let $-\Delta_1(t)$ be the v
the x's are replaced by p's and the y's by q's, so t
) bordered by p's and q's.* Thus

$$S(t)\left[A-(c+t)B\right]T(v)=-\Delta_0(t)\,\Delta_1(v),$$

$$S(t)\left[A-(c+v)B\right]T(v)=-\Delta_0(v)\,\Delta_1(t),$$

now write
$$S(t)/\Delta_1(t)=F(t),$$
$$T(v)/\Delta_1(v)=G(v),$$

e we may observe that F will contain no power of
ilar remark applies to G. Hence

$$F(t)\left[A-(c+t)B\right]G(v)=-\Delta_0(t)/\Delta_1(t),$$

$$F(t)\left[A-(c+v)B\right]G(v)=-\Delta_0(v)/\Delta_1(v)$$

$$(t-v)\,F(t)\,BG(v)=\frac{\Delta_0(t)}{\Delta_1(t)}-\frac{\Delta_0(v)}{\Delta_1(v)},$$

and let us suppose that we have

$$\Delta_0(t)/\Delta_1(t) = t^e(\beta_1+\beta_2 t+\ldots+\beta_e t^{e-1}+\ldots).$$

The quantities $\beta_1, \beta_2, \ldots, \beta_e, \ldots$ will be functions of the coefficients (a, b) and of the arbitrary p's and q's. Since there are $2n$ such arbitraries we shall always have enough constants to satisfy the conditions

$$\beta_2 = 0 = \beta_3 = \ldots = \beta_e,$$

and

$$\beta_1 \neq 0.$$

Then the result of dividing the difference

$$\frac{\Delta_0(t)}{\Delta_1(t)} - \frac{\Delta_0(v)}{\Delta_1(v)}$$

by $(t-v)$ will contain the term

$$\beta_1(t^{e-1}+t^{e-2}v+\ldots+v^{e-1}),$$

and no other term in which either t or v is not raised to a higher power than $(e-1)$; for the next term in the quotient will be

$$\beta_{e+1}(t^{2e-1}+\ldots+v^{2e-1}),$$

in which the terms which most nearly satisfy the condition are $t^e v^{e-1}$ and $t^{e-1}v^e$.

Hence we have, if we take the first e terms of $F(t)$ and $G(v)$,

$$(F_1+F_2t+\ldots+F_et^{e-1})\,B\,(G_1+G_2v+\ldots+G_ev^{e-1}) = \beta_1(t^{e-1}+\ldots+v^{e-1}),$$

and, in exactly the same way,

$$(F_1+F_2t+\ldots+F_et^{e-1})(A-cB)(G_1+G_2v+\ldots+G_ev^{e-1})$$
$$= \beta_1(t^{e-1}v+\ldots+tv^{e-1}).$$

Comparing corresponding terms, we have

$$F_rBG_s = 0 \quad (r+s \neq e+1),$$
$$F_rBG_s = \beta_1 \quad (r+s = e+1),$$
$$F_r(A-cB)\,G_s = 0 \quad (r+s \neq e+2),$$
$$F_r(A-cB)\,G_s = \beta_1 \quad (r+s = e+2),$$

where *of course neither* r nor s can exceed e.

Combining the results just written, we have the three cases

$$F_r(A-\lambda B)\,G_s = (c-\lambda)\,\beta_1, \quad \text{if} \quad r+s=e+1.$$

$$F_r(A-\lambda B)\,G_s = \beta_1, \qquad\quad \text{if} \quad r+s=e+2.$$

$$F_r(A-\lambda B)\,G_s = 0^{*}$$

for any other combination.

Next suppose that $H(v)$ is formed in the same way as $G(v)$, but from another root of Δ_0, say from $\lambda = c'$. Then

$$F(t)\big[A-(c+t)B\big]H(v) = -t^e(\beta+\gamma t+\dots),$$

$$F(t)\big[A-(c'+v)B\big]H(v) = -v^f(\beta'+\gamma' v+\dots),$$

and thus, as $c \neq c'$, we have

$$(F_1+F_2t+\dots+F_et^{e-1})\,B\,(H_1+H_2v+\dots+H_fv^{f-1}) = 0,$$

$$(F_1+\dots \qquad +F_et^{e-1})\,A\,(H_1+\dots \qquad +H_fv^{f-1}) = 0,$$

Thus $$F_r(A-\lambda B)\,H_s = 0 \quad (r \leqq e,\ s \leqq f).$$

If now, we write

$$P_1 = \sum_{r=1}^{e} x_r F_r, \qquad Q_1 = \sum_{r=1}^{e} y_r G_r,$$

with P_2, Q_2, ... similar forms derived from the other roots of

$$\Delta_0 = |\,A-\lambda B\,| = 0,$$

we shall have

$$P_1(A-\lambda B)\,Q_1 = \beta\,(c-\lambda)(x_1y_e+x_2y_{e-1}+\dots+x_ey_1)$$
$$+\beta\,(x_2y_e+x_3y_{e-1}+\dots+x_ey_2),$$

while $$P_1(A-\lambda B)\,Q_2 = 0 = P_2(A-\lambda B)\,Q_1.$$

Hence, if $$P = P_1+P_2+\dots, \quad Q = Q_1+Q_2+\dots,$$

$P(A-\lambda B)\,Q$ will be the sum of a number of groups of terms like those just written.

---

[*] Results similar to these are given by Frobenius in a paper on the skew invariants of bilinear and quadratic forms (*Crelle*, 1879, Bd. LXXXVI., p. 44, § 6). His relate to the expansion of $S(t)/\Delta_0(t)_r$ and are not quite so general as mine; *they give* $F_r(A-\lambda B)\,G_s = 0$, only if $r+s<e+1$.

But in general (unless $h_1 = 0$ for each root) the form $P(Q)$ will not contain n x's (y's), and so we proceed to show how to find similar terms from each successive group of minors.

Take next the bilinear form R_k,

$$\left|\begin{array}{cccccc} a_{11}-\lambda b_{11}, & \dots, & a_{1n}-\lambda b_{1n}, & q_1', & \dots, & q_1^{k-1}, & y_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1}-\lambda b_{n1}, & \dots, & a_{nn}-\lambda b_{nn}, & q_n', & \dots, & q_n^{k-1}, & y_n \\ p_1', & \dots, & p_n', & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_1^{k-1}, & \dots, & p_n^{k-1}, & 0, & \dots, & 0, & 0 \\ x_1, & \dots, & x_n, & 0, & \dots, & 0, & 0 \end{array}\right|,$$

in which the p's and q's are arbitrary constants, and the indices do not represent powers, but are additional suffixes, when applied to the p's and q's. Δ_k will denote a corresponding determinant with $p_1^k, \dots, p_n^k$ in place of $x_1, \dots, x_n$, and $q_1^k, \dots, q_n^k$ in place of $y_1, \dots, y_n$.

Let us suppose that $(\lambda - c)$ to the power h_k is a factor of *every* k^{th} minor of $| A - \lambda B |$, so that this is also a factor of R_k and Δ_k; and we write

$$e_k = h_{k-1} - h_k,$$

so that e_k is the index of the k^{th} invariant-factor* of $| A - \lambda B |$ which belongs to the base $(\lambda - c)$. It will be convenient to denote by $R_k(t)$, $\Delta_k(t)$ the values of R_k, Δ_k, when $\lambda = c + t$.

Consider the symbolical product

$$R_k(t)\left[A - (c+t)B\right];$$

this is equal to the result of substituting for y_r in $R_k(t)$

$$\frac{\partial}{\partial x_r}\left[A - (c+t)B\right];$$

subtract from the last column of the determinant so formed y_1 times the first, y_2 times the second, ..., y_n times the n^{th}. We then see that the product can be written in the form

$$-\Delta_{k-1}(t)E + X_1(p_1'y_1 + \dots + p_n'y_n) + \dots + X_{k-1}(p_1^{k-1}y_1 + \dots + p_n^{k-1}y_n),$$

---

* *Elementartheiler* of Weierstrass.

where $X_1, \dots, X_{k-1}$ are linear functions of the k^{th} minors of $|A-\lambda B|$ and of the x's.

Let us now multiply by $R_k(v)$, and we find

$$R_k(t)\left[A-(c+t)B\right]R_k(v) = -\Delta_{k-1}(t)\,R_k(v),$$

for the product of the remaining terms is found by replacing the x's in $R_k(v)$ by the quantities

$$(p_1' X_1 + \dots + p_1^{k-1} X_{k-1}),\ \dots,\ (p_n' X_1 + \dots + p_n^{k-1} X_{k-1}),$$

and is accordingly zero.

Similarly we have

$$R_k(t)\left[A-(c+v)B\right]R_k(v) = -\Delta_{k-1}(v)\,R_k(t)$$

Suppose, now, that $S_k(t)$ is the value of $R_k(t)$, when $x_1, \dots, x_n$ are replaced by $p_1^k, \dots, p_n^k$; and $T_k(v)$ that of $R_k(v)$, when $y_1, \dots, y_n$ are replaced by $q_1^k, \dots, q_n^k$. Then our two equations give

$$S_k(t)\left[A-(c+t)B\right]T_k(v) = -\Delta_{k-1}(t)\,\Delta_k(v),$$

$$S_k(t)\left[A-(c+v)B\right]T_k(v) = -\Delta_{k-1}(v)\,\Delta_k(t),$$

for in the bilinear forms on the right both x's and y's must be replaced by constants, which gives Δ_k instead of R_k.

We now write

$$S_k(t)/\Delta_k(t) = F_1 + F_2 t + F_3 t^2 + \dots,$$

$$T_k(v)/\Delta_k(v) = G_1 + G_2 v G_3 + v^2 + \dots,$$

and
$$\Delta_{k-1}(t)/\Delta_k(t) = t^e(\beta_1 + \beta_2 t + \dots + \beta_e t^{e-1} + \dots)$$

$$(e = e_k = h_{k-1} - h_k).$$

Thus we find that (if, as before, we choose $p_1^k, \dots, p_n^k$ and $q_1^k, \dots, q_n^k$ so that $\beta_2 = 0 = \beta_3 = \dots = \beta_e$, but $\beta_1 \neq 0$)

$$(F_1 + F_2 t + \dots + F_e t^{e-1})\,B\,(G_1 + G_2 v + \dots + G_e v^{e-1})$$
$$= \beta_1(t^{e-1} + t^{e-2}v + \dots + v^{e-1}),$$

$$(F_1 + F_2 t + \dots + F_e t^{e-1})(A - cB)(G_1 + G_2 v + \dots + G_e v^{e-1})$$
$$= \beta_1(t^{e-1}v + \dots + v^{e-1}t).$$

Hence, as above, we deduce

$$F_r(A-\lambda B)\,G_s = (c-\lambda)\,\beta_1, \quad \text{if} \quad r+s=e+1,$$

$$F_r(A-\lambda B)\,G_s = \beta_1, \quad \text{if} \quad r+s=e+2,$$

$$F_r(A-\lambda B)\,G_s = 0^*$$

for any other combination of r, s.

In the next place suppose that the G's are similarly derived from $\Delta_l(v)$, where let us suppose that $l>k$; then the product

$$R_k(t)\left[A-(c+t)B\right]R_l(v) = -\Delta_{k-1}(t)\,R_l(v).$$

Now we have to replace the x's in $R_k(t)$ with $p_1^k, \ldots, p_n^k$, in order to obtain $S_k(t)$; the same substitution must be made in $R_l(v)$ on the right; and the result will then be zero, for we obtain a determinant in which two rows are identical.

Hence $$S_k(t)\left[A-(c+t)B\right]T_l(v) = 0.$$

Forming the product

$$S_k(t)\left[A-(c+v)B\right]T_l(v),$$

it will be seen to be of the form $\sum\limits_r Z_k^r(t)\,Z_{l-1}^r(v)$, where $Z_k^r(t)$ is a linear function of the k^{th} minors of $|\,A-(c+t)B\,|$, and $Z_{l-1}^r(v)$ of the $(l-1)^{\text{th}}$ minors of $|\,A-(c+v)B\,|$. It follows that

$$\frac{S_k(t)}{\Delta_k(t)}\left[A-(c+v)B\right]\frac{T_l(v)}{\Delta_l(v)} = (t-v)\,\frac{S_k(t)}{\Delta_k(t)}\,[B]\,\frac{T_l(v)}{\Delta_l(v)}$$

is divisible by v^f $(f=e_l)$.

Consequently, if

$$S_k(t)/\Delta_k(t) = F_1 + \ldots \quad F_e t^{e-1} + \ldots,$$

$$T_l(v)/\Delta_l(v) = G_1 + \ldots + G_f v^{f-1} + \ldots,$$

we have $$(F_1 + \ldots + F_e t^{e-1})\,B\,(G_1 + \ldots + G_f v^{f-1}) = 0,$$

---

* See Frobenius (*Crelle*, 1879, Bd. LXXXVI., p. 44, § 7). His results relate to the *expansion of* $S_k(t)/\Delta_{k-1}(t)$, and then $F_r(A-\lambda B)\,G_s = 0$, only if $r+s<e+1$.

and thus also

$$(F_1 + \dots + F_e t^{e-1})(A - \lambda B)(G_1 + \dots + G_f v^{f-1}) = 0,$$

which gives $$F_r\,(A - \lambda B)\,G_s = 0 \quad (r \leqq e_k,\ s \leqq e_l).$$

If the G's be derived from a different root of $| A - \lambda B | = 0$, this equation will also hold; a result which may be deduced by applying the arguments previously used for this case, when the F's and G's were derived from first minors of $| A - \lambda B |$.

If, then, we write $$P = \Sigma x_r F_r,$$
$$Q = \Sigma y_r G_r,$$

where the summation extends to every root of $| A - \lambda B | = 0$, and to all the forms deduced as above from every minor in which that root appears, we shall have

$$P\,(A - \lambda B)\,Q$$

as the sum of a number of groups of the type

$$\beta\,(c - \lambda)(x_1 y_e + x_2 y_{e-1} + \dots + x_e y_1) + \beta\,(x_2 y_e + x_3 y_{e-1} + \dots + x_e y_2).$$

Here e is the index of the power of $(\lambda - c)$ in the corresponding invariant-factor.

The number of x's introduced by the root $\lambda = c$ will be

$$(h_0 - h_1) + (h_1 - h_2) + \dots = h_0,$$

and so the total number of x's in P will be n, as it ought to be.

We may state our result in another form—

Take the minors of q_r^k, p_r^k in $\Delta_k(t)$, and divide each by $\Delta_k(t)$; expand the quotients in powers of t to e_k terms, and replace the powers of t by different x's in the former, by y's in the latter. Finally, sum the similar results for all roots of $| A - \lambda B | = 0$, and for every order k of $\Delta_k(t)$; we thus arrive at the linear substitutions for x_r, y_r respectively. When all the variables in A, B are subjected to the corresponding substitutions, the forms A, B will be reduced to the canonical forms

$$\Sigma c\beta\,(x_1 y_e + \dots + x_e y_1) + \Sigma\beta\,(x_2 y_e + \dots + x_e y_2),$$

$$\Sigma\beta\,(x_1 y_e + \dots + x_e y_1), \quad \dots$$

respectively.

If $| B | = 0$, one or more of the roots of the equation

$$| A - \lambda B | = 0$$

will be infinite. Suppose that infinite roots are present also in the minors of $| A-\lambda B |$; in fact, take h_k to be the highest index of λ present in Δ_k; then there will be $(n-k-h_k)$ infinite roots in every k^{th} minor of $| A-\lambda B |$; if $h_k = n-k$, there will be no infinite root, and, of course, h_k is not greater than $(n-k)$.

Just as before, we have

$$S_k(t)\,[A-B/t]\,T_k(v) = -\Delta_{k-1}(t)\,\Delta_k(v),$$

$$S_k(t)\,[A-B/v]\,T_k(v) = -\Delta_{k-1}(v)\,\Delta_k(t),$$

where now $S_k(t)$, $\Delta_k(t)$, ... denote the values of S_k, Δ_k, ..., with $\lambda = 1/t$.

Thus we have

$$\frac{S_k(t)}{\Delta_k(t)}\left[At-B\right]\frac{T_k(v)}{\Delta_k(v)} = -t\,\frac{\Delta_{k-1}(t)}{\Delta_k(t)},$$

$$\frac{S_k(t)}{\Delta_k(t)}\left[Av-B\right]\frac{T_k(v)}{\Delta_k(v)} = -v\,\frac{\Delta_{k-1}(v)}{\Delta_k(v)}.$$

If $t\Delta_{k-1}(t)/\Delta_k(t)$ be divisible by $t^e\,(e = e_k)$, it will follow that there are $e_k\,(= 1+h_k-h_{k-1})$ more infinite roots of Δ_{k-1} than there are of Δ_k; or e_k has the same meaning as that usually attached to this symbol in connexion with finite roots.

Now let us write

$$\frac{S_k(t)}{\Delta_k(t)} = F_1+F_2t+\ldots+F_et^{e-1}+\ldots,$$

$$\frac{T_k(v)}{\Delta_k(v)} = G_1+G_2v+\ldots+G_ev^{e-1}+\ldots,$$

$$t\,\frac{\Delta_{k-1}(t)}{\Delta_k(t)} = t^e(\beta_1+\beta_2t+\ldots+\beta_et^{e-1}+\ldots);$$

then the equations above will yield

$$F_r(A-\lambda B)\,G_s = 0, \quad \text{unless} \quad (r+s) = (e+1) \quad \text{or} \quad (e+2);$$

and $$F_r(A-\lambda B)\,G_s = -\beta_1, \quad \text{if} \quad (r+s) = (e+1),$$

$$F_r(A-\lambda B)\,G_s = +\lambda\beta_1, \quad \text{if} \quad (r+s) = (e+2),$$

provided that $p_1^k, \ldots, p_n^k$ and $q_1^k, \ldots q_n^k$ are chosen so as to satisfy the conditions

$$\beta_2 = \beta_3 = \ldots = \beta_e = 0, \quad \beta_1 \neq 0.$$

Exactly as before, we show that

$$F_r\,(A-\lambda B)\;G_s = 0,$$

if G_s be derived from minors of an order different from that to which F_r belongs; or if G_s be derived from a root of $|\,A-\lambda B\,| = 0$ which is not infinite.

If, then, we have terms in P, Q of the form $\Sigma x_r F_r$, $\Sigma y_r G_r$ given by the infinite roots of $|\,A-\lambda B\,| = 0$, the corresponding parts of $P\,(A-\lambda B)\;Q$ will be of the form

$$\Sigma\lambda\beta\,(x_2 y_e + \ldots + x_e y_2) - \Sigma\beta\,(x_1 y_e + \ldots + x_e y_1).$$

It may be useful to point out that this result is equivalent to what would have been obtained by taking the equation

$$|\,\lambda A - B\,| = 0$$

and working with the corresponding zero roots. It may also be pointed out that we are not left with arbitraries g, h, as in Weierstrass's reduction of this case; Weierstrass's investigation[*] depends on expanding a certain function in descending powers of λ, in two distinct ways and then equating coefficients; in case $|\,B\,| = 0$, one form of the expansion does not take the same shape as it does in the general case. In this way it became necessary to replace B by $(gA + hB)$, where

$$|\,gA + hB\,| \neq 0.$$

Darboux's corresponding reduction is left without g, h, and this is due to the fact that Darboux does not expand in powers of $(1/\lambda)$, but follows a slightly different method for finding the reduced forms.[†]

According to an extension of Stickelberger's theorem to be found above (p. 86), it will be seen that the forms $F_1, F_2, \ldots, F_e$ $(e = e_k)$ will contain h_{k-1} effective arbitrary constants; and the same number will appear in $G_1, \ldots, G_e$, but these have to satisfy e_k relations, and thus the effective number is

$$2h_{k-1} - e_k = h_{k-1} + h_k.$$

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[*] *In the second paper* quoted above; or see Muth's *Elementartheiler*, pp. 78-81.

[†] *See* § IX. *of* Darboux's paper.

Thus, in all, we obtain

$$h_0 + 2h_1 + 2h_2 + \dots,$$

corresponding to any given root; this number is also equal to

$$\begin{aligned} &e_1 + e_2 + e_3 + \dots + e_k \\ &+2(e_2 + e_3 + \dots + e_k) \\ &\quad +2(e_3 + \dots + e_k) \\ &+ \quad \dots \quad \dots \quad \dots \\ &\qquad\qquad + 2e_k \\ &= e_1 + 3e_2 + 5e_3 + \dots + (2k-1)\,e_k, \end{aligned}$$

which agrees with the numbers given by Frobenius* and Landsberg.†

We come next to the consideration of our problem in case $| A - \lambda B | \equiv 0$ for all values of λ. In order to be perfectly general, I shall suppose that $\Delta_0 \equiv 0 \equiv \Delta_1 \equiv \Delta_2 \equiv \dots \equiv \Delta_{k-1}$, and that Δ_k is not identically zero. Then I have shown in Part 1 of this paper that Δ_k can be arranged as the product of three determinants D_1, D_2, D_3; ‡ where D_1 consists of k rows and columns containing only p's, D_2 consists of k rows and columns containing only q's, and D_3 contains neither p's nor q's. It has also been proved that by proper choice of the p's and q's we can make D_1 and D_2 each unity; and corresponding to each set of p's (say $p_1^r, \dots, p_n^r$) there is a set of π's $(\pi_1^r, \dots, \pi_n^r)$. The π's have the property that, if $(\pi_1^r, \dots, \pi_n^r)$ take the place of $(p_1^r, \dots, p_1^n)$ in D_1, then the value of D_1 is λ^a ($a = a_r$, one of the *Minimalgradzahlen*); but, if $(\pi_1^r, \dots, \pi_n^r)$ take the place of any other set of p's in D_1, then the resulting determinant is identically zero. There are in the same way sets of κ's, one corresponding to each set of q's, and possessing properties similar to those of the π's.

We denote by R_r $(r = 1, 2, \dots, k)$ the bilinear form obtained from Δ_k by replacing $(p_1^r, \dots, p_n^r)$ with $(x_1, \dots, x_n)$ and $(q_1^r, \dots, q_n^r)$ with $(y_1, \dots, y_n)$. Consider the value of the symbolical product $R_r(A - \lambda B)$; which is obtained by substituting for $(y_1, \dots, y_n)$ in R_r the values

$$\left(\frac{\partial A}{\partial x_1} - \lambda\frac{\partial B}{\partial x_1}, \dots, \frac{\partial A}{\partial x_n} - \lambda\frac{\partial B}{\partial x_n}\right).$$

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* *Crelle*, 1878, Bd. LXXXIV., p. 29.

† *Ibid.*, 1896, Bd. CXVI., p. 349.

‡ *Called there* E_1, E_2, E_3 (pp. 88 *et seq.*).

Subtract from the column so obtained y_1 times the first column, y_2 times the second, ..., y_n times the n^{th}; we then observe that the resulting determinant is a linear function of the $(k-1)^{\text{th}}$ minors of $|A-\lambda B|$, and is therefore identically zero. Thus

$$R_r(A-\lambda B) = 0 = (A-\lambda B) R_r.$$

by a similar investigation.

Again, let us denote by U the bilinear form obtained by bordering Δ_k with $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$. Then $(A-\lambda B)U$ is found by substituting for $(x_1, \dots, x_n)$ in U the values

$$\left(\frac{\partial A}{\partial y_1} - \lambda\frac{\partial B}{\partial y_1}, \dots, \frac{\partial A}{\partial y_n} - \lambda\frac{\partial B}{\partial y_n}\right);$$

now multiply the first row of the resulting determinant by x_1, the second by x_2, ..., the n^{th} by x_n, and subtract from the last row. Then

$$(A-\lambda B)U = \begin{vmatrix} a_{11}-\lambda b_{11}, & \dots, & a_{1n}-\lambda b_{1n}, & q_1', & \dots, & q_1^k, & y_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1}-\lambda b_{n1}, & \dots, & a_{nn}-\lambda b_{nn}, & q_n', & \dots, & q_n^k, & y_n \\ p_1', & \dots, & p_n', & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p_1^k, & \dots, & p_n^k, & 0, & \dots, & 0, & 0 \\ 0, & \dots, & 0, & -\xi_1, & \dots, & -\xi_k, & -E \end{vmatrix},$$

in which we have written

$$\xi_r = q_1^r x_1 + \dots + q_n^r x_n \quad (r = 1, 2, \dots, k),$$

and, as usual, $$E = x_1 y_1 + \dots + x_n y_n.$$

Expanding the determinant, we have

$$(A-\lambda B)U = -\Delta_k E + \sum_{r=1}^{k} \xi_r \Psi_r,$$

where we have written Ψ_r to denote the linear function derived from Δ_k by replacing $(q_1^r, \dots, q_n^r)$ with $(y_1, \dots, y_n)$. It will be convenient to use Φ_r to denote a similar linear function with $(x_1, \dots, x_n)$ in place of $(p_1^r, \dots, p_1^n)$ in Δ_k.

We shall require in what follows to distinguish between certain of our bilinear and linear forms according as they are derived from $(A-\lambda B)$ or $(A-\mu B)$; this will be indicated by writing λ or μ in brackets; thus $U(\mu)$ will be a bilinear form derived from $(A-\mu B)$ in the same way as $U(\lambda)$ (written U above) is derived from $(A-\lambda B)$.

We proceed to consider the symbolical product

$$R_r(\lambda)(A-\mu B)\,U(\mu) = -\Delta_k(\mu)\,R_r(\lambda)+\sum_{s=1}^{k} R_r(\lambda)\left[\xi_s\Psi_s(\mu)\right],$$

where the quantity in square brackets is a bilinear form to be combined with $R_r(\lambda)$ in the usual way. The product $R_r(\lambda)\left[\xi_s\Psi_s(\mu)\right]$ is found by replacing $(y_1, \ldots, y_n)$ in $R_r(\lambda)$ by the quantities

$$\left[q_1^s\Psi_s(\mu), \ldots, q_n^s\Psi_s(\mu)\right];$$

the resulting determinant has two rows the same, unless $r = s$, and so the product is in general zero. Also we see that

$$R_r(\lambda)\left[\xi_r\Psi_r(\mu)\right] = \Phi_r(\lambda)\,\Psi_r(\mu),$$

and thus $R_r(\lambda)(A-\mu B)\,U(\mu) = -\Delta_k(\mu)\,R_r(\lambda)+\Phi_r(\lambda)\,\Psi_r(\mu)$.

In this result let us replace the final x's by $p_1^r, \ldots, p_n^r$ and the final y's by $\kappa_1^s, \ldots, \kappa_n^s$; then in the symbolical product on the left only the x's of $R_r(\lambda)$ and the y's of $U(\mu)$ are to undergo this substitution. Hence, if $W_s(\mu)$ be the form now found for $U(\mu)$,

$$\Phi_r(\lambda)(A-\mu B)\,W_s(\mu) = 0 \qquad (r \neq s),$$

since we have that $R_r(\lambda)$, $\Psi_r(\mu)$ both vanish if the y's are replaced by $\kappa_1^s, \ldots, \kappa_n^s$—a result following by the properties of the κ's. But when the y's are replaced by $\kappa_1^r, \ldots, \kappa_n^r$ and the x's by $p_1^r, \ldots, p_n^r$ we have

$$\left.\begin{aligned} R_r(\lambda) &= \lambda^\beta\Delta_k(\lambda) \\ \Phi_r(\lambda) &= \Delta_k(\lambda) \\ \Psi_r(\mu) &= \mu^\beta\Delta_k(\mu) \end{aligned}\right\} \quad (\beta = \beta_r).$$

So we now have the equation

$$\Phi_r(\lambda)(A-\mu B)\,W_r(\mu) = \Delta_k(\lambda)\,\Delta_k(\mu)\,(\mu^\beta-\lambda^\beta);$$

let us write then

$$\left.\begin{aligned} \Phi_r(\lambda)/\Delta_k(\lambda) &= F_1+F_2\lambda+\ldots+F_{\beta+1}\lambda^\beta \\ W_r(\mu)/\Delta_k(\mu) &= G_1+G_2\mu+\ldots+G_\beta\mu^{\beta-1}+\ldots \end{aligned}\right\}\text{[*]} \quad (\beta = \beta_r).$$

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[*] We observe that $\Phi_r(\lambda)$ will break up into factors, two of which are the same as in $\Delta_k(\lambda)$; the third will be a function of degree β_r in λ; while the third factor of $\Delta_k(\lambda)$ is unity, *by the definition* of our q's.

Thus

$$(F_1+F_2\lambda+\ldots+F_{\beta+1}\lambda^{\beta})(A-\mu B)(G_1+G_2\mu+\ldots+G_{\beta}\mu^{\beta-1}+\ldots)=\mu^{\beta}-\lambda^{\beta},$$

and further, since $\quad R_r(\lambda)(A-\lambda B)=0,$

we have

$$(F_1+F_2\lambda+\ldots+F_{\beta+1}\lambda^{\beta})(A-\lambda B)(G_1+G_2\mu+\ldots+G_{\beta}\mu^{\beta-1}+\ldots)=0.$$

Hence $\quad F_{\rho}BG_{\sigma}=0 \quad$ unless $\quad \rho+\sigma=\beta_r+1,$

$\quad F_{\rho}BG_{\sigma}=-1 \quad$ if $\quad \rho+\sigma=\beta_r+1,$

and $\quad F_{\rho}AG_{\sigma}=-1 \quad$ if $\quad \rho+\sigma=\beta_r+2,$

$\quad F_{\rho}AG_{\sigma}=0 \quad$ unless $\quad \rho+\sigma=\beta_r+2.$

If, however, the G's are derived from $W_s(\mu)$ and not from $W_r(\mu)$, we shall have

$$F_{\rho}BG_{\sigma}=0=F_{\rho}AG_{\sigma}$$

for all values of ρ, σ.

We can, in the same way, determine more F's and G's by interchanging the parts played by R_r and U; and then the β's will be replaced by α's. We must also consider the value of products such as

$$U(\lambda)(A-\mu B)\,U(\mu)=-\Delta_k(\mu)\,U(\lambda),$$

for the product $U(\lambda)\left[\xi_s\Psi_s(\mu)\right]$ is always zero. Similarly,

$$U(\lambda)(A-\lambda B)\,U(\mu)=-\Delta_k(\lambda)\,U(\mu).$$

Let us write $V_r(\lambda)$ for the value of $U(\lambda)$ with $x_1, \ldots, x_n$ replaced by $\pi_1^r, \ldots, \pi_n^r$; and, as before, $W_s(\lambda)$ for its value with $y_1, \ldots, y_n$ replaced by $\kappa_1^s, \ldots, \kappa_n^s$. Finally, let $U_{rs}(\lambda)$ be its value when both substitutions are made; so that

$$\frac{V_r(\lambda)}{\Delta_k(\lambda)}(A-\lambda B)\frac{W_s(\mu)}{\Delta_k(\mu)}=-\frac{U_{rs}(\mu)}{\Delta_k(\mu)},$$

and

$$\frac{V_r(\lambda)}{\Delta_k(\lambda)}(A-\mu B)\frac{W_s(\mu)}{\Delta_k(\mu)}=-\frac{U_{rs}(\lambda)}{\Delta_k(\lambda)}.$$

Now, in determining $p^r, \ldots, p_n^r$ as above, we are left with $(n-\Sigma\alpha_r-k)$ constants at our disposal; this number is also equal to $(\gamma+\Sigma\beta_r)$, if we write γ to denote the index of the highest power of λ in D_s.* We have the same number of constants left in finding $\pi_1^r, \ldots, \pi_n^r$. So, in all, from the k sets of p, π, q, κ, we have

$$2k(\gamma+\Sigma\beta+\gamma+\Sigma\alpha)=2k(n+\gamma-k)$$

[* *The highest power of* λ in Δ_k is $(n-k)$ and is also $\Sigma\alpha_r+\Sigma\beta_r+\gamma$.—*Sept. 14th,* 1900.]

constants left at our disposal. We can choose these constants so that the expansion of $U_{rs}(\lambda)/\Delta_k(\lambda)$ contains no terms after the term independent of λ, until we reach the $(a_r+\beta_s+1)^{\text{th}}$ power of λ; for, in all, this requires us to satisfy $\Sigma\Sigma(a_r+\beta_s)$ conditions; and this number is $k(n-k-\gamma)$. Hence

$$(\lambda-\mu)\frac{V_r(\lambda)}{\Delta_k(\lambda)}B\frac{W_s(\mu)}{\Delta_k(\mu)} = K(\lambda^{\theta+1}-\mu^{\theta+1})+\dots \quad (\theta = a_r+\beta_s).$$

But, on expansion, we stop $V_r(\lambda)/\Delta_k(\lambda)$ at the term in λ^{a-1} $(a = a_r)$, and $W_s(\mu)/\Delta_k(\mu)$ at $\mu^{\beta-1}$ $(\beta = \beta_s)$. Thus

$$(F_1+F_2\lambda+\dots+F_a\lambda^{a-1})B(G_1+G_2\mu+\dots+G_\beta\mu^{\beta-1}) = 0,$$

or $$F_\rho BG_\sigma = 0 \quad (\rho \overset{=}{<} a_r,\ \sigma \overset{=}{<} \beta_s).$$

Similarly, $$F_\rho AG_\sigma = 0$$

for the same range of values of ρ, σ.

Take now the two forms $P = \Sigma x_r F_r$, $Q = \Sigma y_r G_r$, where the summation is to extend to all F's and G's just calculated; so that the number of x's contained in P is

$$\Sigma a+\Sigma(\beta+1) = k+\Sigma a+\Sigma\beta = (n-\gamma);$$

and in the same way Q contains $(n-\gamma)$ y's. Then the reduced form

$$\begin{aligned}P(A-\lambda B)Q = {} & \Sigma\left[(\lambda x_1-x_2)y_\beta+(\lambda x_2-x_3)y_{\beta-1}+\dots+(\lambda x_\beta-x_{\beta+1})y_1\right]\\ & +\Sigma\left[(\lambda\eta_1-\eta_2)\xi_a+(\lambda\eta_2-\eta_3)\xi_{a-1}+\dots+(\lambda\eta_a-\eta_{a+1})\xi_1\right].\end{aligned}$$

In order to complete the expressions for our substitutions, we must next return to $\Delta_k(\lambda)$; this will contain γ factors linear in λ. We can proceed with these just as if $\Delta_k(\lambda)$ were the original determinant $|A-\lambda B|$ of our former work; we shall thus obtain γ more F's, and the same number of G's. So that our final expressions for P, Q contain n x's and n y's, as they should; and it is easy to see that the additional terms in $P(A-\lambda B)Q$ will be of the types found at the beginning of this investigation.

As a numerical illustration and to explain the process, we take the following case, suggested by an example given by Darboux[*]:—

Let $$A = (ax_1y_1+bx_2y_2+cx_4y_4)+x_2y_1+x_3y_2+x_4y_3,$$

$$B = x_1y_1+x_2y_2+x_4y_4.$$

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[*] On p. 395 of the paper already quoted; see also p. 117 below.

It is then easy to show that the (general) value of $\Delta_1(\lambda)$ will be

$$\left[p_3(\lambda-c)+p_4\right]\left[q_1+q_2(\lambda-a)+q_3(\lambda-a)(\lambda-b)\right].^*$$

The conditions specified for the p's and q's will give

$$p_3=0,\quad p_4=1,$$
$$\pi_3=1,\quad \pi_4=c,$$
$$q_1=1,\quad q_2=0,\quad q_3=0,$$
$$\kappa_1=a^2,\quad \kappa_2=a+b,\quad \kappa_3=1.$$

These yield

$$\begin{aligned}\Phi_1(\lambda)&=y_1+y_2(\lambda-a)+y_3(\lambda-a)(\lambda-b),\\ \Psi_1(\lambda)&=\qquad\qquad x_3(\lambda-c)\qquad +x_4.\\ \Delta_1(\lambda)&=1.\end{aligned}$$

We next form $U_{11}(\lambda)$, which is $\Delta_1(\lambda)$, bordered with the π's and κ's; and, on calculation, this is found to be

$$\kappa_4-\lambda^2 q_4+\lambda\left[p_2-(a+\lambda)\,p_1\right],$$

for our previous conditions have not determined p_1, p_2, π_1, π_2, q_4, κ_4; it follows that $U_{11}(\lambda)\equiv 0$, if we take $p_1=0$, $p_2=0$, $\kappa_4=0$, $q_4=0$. We have nothing to determine π_1, π_2, but these may be assumed to be zero, in order to simplify the reduction as far as possible.

We have then

$$\begin{aligned}V_1(\lambda)&=\qquad\qquad y_4,\\ W_1(\lambda)&=x_1(a+\lambda)+x_2.\end{aligned}$$

So, on the whole, we get the substitutions

$$\left.\begin{aligned}x_1&=\xi_1\\ x_2&=-a\xi_1+\xi_2\\ x_3&=\ ab\xi_1-(a+b)\,\xi_2+\xi_3\end{aligned}\right\}\ \text{derived from } \Phi_1(\lambda),$$

$$x_4=\qquad\qquad \xi_1' \quad \text{from } V_1(\lambda),$$

and

$$\left.\begin{aligned}y_1&=\qquad a\eta_1'+\eta_2'\\ y_2&=\qquad \eta_1'\end{aligned}\right\}\ \text{from } W_1(\lambda),$$

$$\left.\begin{aligned}y_3&=-c\eta_1+\eta_2\\ y_4&=\quad \eta_1\end{aligned}\right\}\ \text{from } \Psi_1(\lambda).$$

[* The sign of $\Delta_1(\lambda)$ should be changed; this will reverse the signs of $\Phi_1(\lambda)$, $\Psi_1(\lambda)$, *and explains* why the forms of p. 117 differ in sign from those on p. 115.—[illegible]*, 1900.*]

Thus, finally, we have

$$\begin{aligned} A &= y_1(ax_1+x_2)+y_2(bx_2+x_3)+x_4(y_3+cy_4) \\ &= \xi_2(a\eta_1'+\eta_2')+(-a\xi_2+\xi_3)\eta_1'+\xi_1'\eta_2 \\ &= \xi_2\eta_2'+\xi_3\eta_1'+\xi_1'\eta_2, \end{aligned}$$

and

$$\begin{aligned} B &= x_1y_1+x_2y_2+x_4y_4 \\ &= \xi_1(a\eta_1'+\eta_2')+(-a\xi_1+\xi_2)\eta_1'+\xi_1'\eta_1 \\ &= \xi_1\eta_2'+\xi_2\eta_1'+\xi_1'\eta_1, \end{aligned}$$

i.e.,

$$A-\lambda B = (\xi_2-\lambda\xi_1)\eta_2'+(\xi_3-\lambda\xi_2)\eta_1'+\xi_1'(\eta_2-\lambda\eta_1),$$

which is the canonical form required, agreeing with our general result on p. 115.

I have attempted to construct a similar special case for which it would be necessary to work with $\Delta_2(\lambda)$; but the work is very laborious unless we start with forms simplified so much that the reducing substitutions are self-evident. I therefore content myself with adding the solution of Darboux's special case; which is

$$\begin{aligned} A &= a(x_1y_1+\ldots+x_{n-1}y_{n-1})+x_2y_1+\ldots+x_ny_{n-1}, \\ B &= \quad x_1y_1+\ldots+x_{n-1}y_{n-1}. \end{aligned}$$

The (general) value of $\Delta_1(\lambda)$ is then found to be

$$p_n\left[q_1+q_2(\lambda-a)+\ldots+q_n(\lambda-a)^{n-1}\right];^*$$

so our special values will be

$$\begin{gathered} p_n=1, \quad \pi_n=1, \\ q_1=1, \quad q_2=0=q_3=\ldots=q_n, \\ \kappa_1=a^{n-1}, \quad \kappa_2=(n-1)a^{n-2}, \quad \kappa_r=\binom{n-1}{r-1}a^{n-r}, \quad \kappa_n=1. \end{gathered}$$

Thus

$$\begin{aligned} \Phi_1(\lambda) &= y_1+y_2(\lambda-a)+\ldots+y_n(\lambda-a)^{n-1}, \\ \Psi_1(\lambda) &= x_n, \end{aligned}$$

and, on further calculation, we find

$$W_1(\lambda) = x_{n-1}+\Sigma\left[\binom{n-2}{r}a^r+\binom{n-3}{r-1}a^{r-1}\lambda+\ldots+\lambda^r\right]x_{n-r-1},$$

in which $p_2, p_3, \ldots, p_n$ have all been put zero, for these values of the p's will make $U_{11}(\lambda)\equiv 0$, if $\pi_1=\pi_2=\ldots=\pi_{n-1}=0$.

[* The sign of $\Delta_1(\lambda)$ should be changed; see footnote on p. 116.—Sept. 14th, 1900.]

Hence we find the substitutions

$$x_1 = \xi_1,$$

$$x_2 = -a\xi_1+\xi_2,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_{r+1} = (-1)^r \left[a^r\xi_1 - ra^{r-1}\xi_2 + \ldots + (-1)^s \binom{r}{s} a^{r-s}\xi_{s+1} + \ldots + (-1)^r \xi_{r+1}\right]$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

all derived from $\Phi_1(\lambda)$; and

$$y_1 = a^{n-2}\eta_1' + a^{n-3}\eta_2' + \ldots + \eta_{n-1}',$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$y_{n-r-1} = \binom{n-2}{r} a^r\eta_1' + \binom{n-3}{r-1} a^{r-1}\eta_2' + \ldots + \eta_{r+1}',$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

derived from $W_1(\lambda)$, and further

$$y_n = \eta_1$$

from $\Psi_1(\lambda)$.

We then have

$$A - \lambda B = (\xi_2 - \lambda\xi_1)\,\eta_{n-1}' + \ldots + (\xi_n - \lambda\xi_{n-1})\,\eta_1',$$

which is the same as Darboux's result. It will be seen that our substitution for the x's agrees with his, but in his work the η's are given in terms of the y's, so that the other substitution cannot be compared directly.

Thursday, March 8th, 1900.

Prof. ELLIOTT, F.R.S., Vice-President, in the Chair.

Fifteen members present.

The Chairman announced that, in accordance with a resolution of the Council, the meetings in future would be held at 5.30 p.m., *instead of at 8 p.m.*

Votes of condolence with the families of the late Prof. Beltrami* and Mr. J. J. Walker were carried in silence, and the Senior Secretary was directed to convey the announcements to the proper quarters.

Prof. Lamb read a paper entitled "Problems relating to the Impact of Waves on a Spherical Obstacle in an Elastic Medium." Dr. Larmor spoke on the subject.

Mr. W. F. Sheppard (Prof. Lamb, Vice-President, in the Chair) spoke on "The Use of Auxiliary Curves in Statistics, with Tables for the Curve of Error." Messrs. Lamb, MacMahon, and Hargreaves joined in a discussion with the author.

A "Supplementary Note on the Theory of Automorphic Functions," by Prof. A. C. Dixon, was taken as read.

The following presents were made to the Library:—

"Educational Times," March, 1900.

"Indian Engineering," Vol. xxvii., Nos. 3–6, Jan. 20–Feb. 10, 1900.

"Transactions of the American Mathematical Society," Vol. i., No. 1, 1900.

"Mathematical Gazette," Vol. i., No. 19.

"Supplemento al Periodico di Matematica," Anno 3, Fasc. iv., Feb., 1900; Livorno.

Bechmann, E.—"Neue Vorrichtungen zum Färben nichtleuchtender Flammen" (Spektrallampen), 8vo; Leipzig, 1900.

De Morgan, A.—"Elementary Illustrations of the Differential and Integral Calculus," 8vo; London, 1899.

"Mathematical Questions and Solutions," Vol. lxxii., edited by D. Biddle, 8vo; London, 1900.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. lxvi., Nos. 424, 425, 1900.

"American Mathematical Society Annual Register"; New York, January, 1900.

"Bulletin des Sciences Mathématiques," Tome xxiii., Dec., 1899; Paris.

"Journal für die reine und angewandte Mathematik," Bd. cxxi., Heft 4; Berlin, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. ix., Fasc. 2, 3, Sem. 1; Roma, 1900.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," No. 6, Bd. li., Math.-Phys. Theil; Allgemeiner Theil, and Naturwissenschaftlicher Theil; 1899.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. xliii., Pt. 5, Vol. xliv., Pt. 1; 1898–1899, 1899–1900.

"Acta Mathematica," Vol. xxiii., Nos. 1, 2; Stockholm, 1899.

*** Copies (two of different dates) of *Il Giorno* and a copy of *La Perseveranza*, sent by Prof. Cremona, were laid on the table, and also a memorial card from the Signora Beltrami.**

Problems relating to the Impact of Waves on a Spherical Obstacle in an Elastic Medium. *By* Prof. HORACE LAMB, F.R.S.

Read and received March 8th, 1900.

The scattering of elastic waves by a spherical obstacle whose properties differ slightly from those of the surrounding medium was discussed long ago, and applied to the illustration of optical phenomena, by Lord Rayleigh.* The present paper takes up the same problem in a more general form; thus in § 5 the obstacle is supposed to be rigid and fixed; in § 6 it is rigid but movable; we next consider the case of a rigid shell containing a vibratory mechanism (§ 7) or a gyrostat (§ 8); in § 9 the effect of a spherical cavity in the medium is examined; finally, in § 10, we have the case of an elastic sphere whose density and rigidity may differ to any extent from those of the surrounding medium.

The investigation was undertaken originally with a view of obtaining a mechanical illustration of the selective absorption of light by a gas, on the hypothesis that this (apparent) absorption is due to a coincidence between the period of the incident waves and that of a free mode of vibration of the molecules. It was found, however, that in this field, as well as in others, the electromagnetic theory furnishes in some respects a much more complete analogy with optical conditions. It has also the advantage of greater mathematical simplicity. For these reasons the calculations of this paper were for a time suspended; they are now completed and published in the hope that they may still present some features of dynamical interest

The point above referred to, having been in the meantime discussed at length in the papers cited below, is here treated more briefly in § 11. It appears that the law of dissipation of radiant energy by scattering (when this is a maximum) has the same form as in the acoustical analogy considered in a former paper,† viz., we find that the ratio (I) which the energy dissipated outwards per unit time from the sphere bears to the energy-flux in the incident waves is given by

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* *Phil. Mag.*, June, 1871; *Scientific Papers*, Vol. I., p. 104.

† "A Problem in Resonance, illustrative of the Mechanical Theory of Selective Absorption of Light by a Gas," *Proc. Lond. Math. Soc.*, Vol. XXXII., p. 11.

$$I = \frac{2n+1}{2\pi}\lambda^2, \tag{1}$$

where λ is the wave-length, and n is the index of the spherical harmonic belonging to the particular type of free vibration in question. It is remarkable that this ratio is independent of the size or constitution of the spherical obstacle, and involves only the wave-length. The same law (as might be expected from the nature of the analysis) is met with in the corresponding electro-magnetic problem,* and appears to have a very wide range of generality.

The notation for the various solutions of the differential equation

$$\frac{d^2F}{d\zeta^2} + \frac{2(n+1)}{\zeta}\frac{dF}{d\zeta} + F = 0 \tag{2}$$

which are required in investigations such as the present can hardly be said to be definitely fixed. In the author's *Hydrodynamics*,† it has been proposed to write

$$\begin{aligned}\psi_n(\zeta) &= (-)^n \left(\frac{d}{\zeta d\zeta}\right)^n \frac{\sin\zeta}{\zeta} \\ &= \frac{1}{1.3\ldots(2n+1)}\left\{1 - \frac{\zeta^2}{2(2n+3)} + \frac{\zeta^4}{2.4(2n+3)(2n+5)} - \ldots\right\},\end{aligned} \tag{3}$$

$$\begin{aligned}\Psi_n(\zeta) &= (-)^n \left(\frac{d}{\zeta d\zeta}\right) \frac{\cos\zeta}{\zeta} \\ &= \frac{1.3\ldots(2n-1)}{\zeta^{2n+1}}\left\{1 - \frac{\zeta^2}{2(1-2n)} + \frac{\zeta^4}{2.4(1-2n)(3-2n)} - \ldots\right\},\end{aligned} \tag{4}$$

and
$$f_n(\zeta) = (-)^n \left(\frac{d}{\zeta d\zeta}\right)^n \frac{e^{-i\zeta}}{\zeta} = \Psi_n(\zeta) - i\psi_n(\zeta)\,; \tag{5}$$

these notations will be adhered to in the present paper. The above functions all satisfy equations of the types

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* *Camb. Trans.*, Vol. XVIII., p. 348. [See also L. Lorenz, *Œuvres Scientifiques*, Copenhagen, 1898, t. I., p. 499, where the formula (1) is arrived at from a somewhat different basis. The investigation was published (in Danish) in 1890; the writer regrets that he has only recently become acquainted with it.—*Sept.*, 1900.]

† §§ 267, 309.

$$\psi_n'(\zeta) = -\zeta\psi_{n+1}(\zeta), \tag{6}$$

$$\zeta\psi_n'(\zeta) + (2n+1)\psi_n(\zeta) = \psi_{n-1}(\zeta), \tag{7}$$

which are frequently required in reductions. We shall also have occasion to appeal to the formula

$$x\phi_n = \frac{r^2}{2n+1}\left(\frac{d\phi_n}{dx} - r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}\right) \tag{8}$$

and its companions, where ϕ_n is a solid harmonic of (positive or negative) degree n.

General Equations.

1. The component displacements (u, v, w) at any point of an incompressible isotropic medium satisfy the equations

$$\frac{d^2u}{dt^2} = c^2\nabla^2 u + \frac{1}{\rho}\frac{dp}{dx}, \quad \frac{d^2v}{dt^2} = c^2\nabla^2 v + \frac{1}{\rho}\frac{dp}{dy}, \quad \frac{d^2w}{dt^2} = c^2\nabla^2 w + \frac{1}{\rho}\frac{dp}{dz}, \tag{9}$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{10}$$

where c is the wave-velocity, connected with the rigidity (μ) and the density (ρ) by the relation

$$c^2 = \mu/\rho, \tag{11}$$

and p denotes the mean normal traction about the point (x, y, z). From these we deduce

$$\nabla^2 p = 0. \tag{12}$$

In the case of simple-harmonic motion we assume a time-factor e^{ikct}, so that $2\pi/k$ denotes the wave-length of plane waves of the same frequency. The equations (9) then take the forms

$$(\nabla^2 + k^2)\,u = -\frac{1}{\mu}\frac{dp}{dx}, \quad (\nabla^2 + k^2)\,v = -\frac{1}{\mu}\frac{dp}{dy}, \quad (\nabla^2 + k^2)\,w = -\frac{1}{\mu}\frac{dp}{dz}. \tag{13}$$

In the problems which we have in view the medium is limited internally by a spherical surface (of radius a) described about the origin, and the disturbance may be regarded as made up of two parts, viz., the extraneous disturbance due to given sources at a distance, and the waves scattered outwards by the sphere.

If the medium were uninterrupted, the displacements due to the extraneous disturbance would be finite at the origin. Hence for the incident wave-system we may write

$$\left.\begin{aligned} p &= -\mu k^2 \Sigma r^n S_n \\ u = \Sigma u_n, \quad v &= \Sigma v_n, \quad w = \Sigma w_n \end{aligned}\right\}, \tag{14}$$

where

$$\left.\begin{aligned} u_n &= \frac{d}{dx} r^n S_n + (n+1)\,\psi_{n-1}(kr)\frac{d}{dx} r^n T_n - nk^2 r^{2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{T_n}{r^{n+1}} \\ &\qquad + \psi_n(kr)\left(y\frac{d}{dz} - z\frac{d}{dy}\right) r^n U_n \\ v_n &= \frac{d}{dy} r^n S_n + (n+1)\,\psi_{n-1}(kr)\frac{d}{dy} r^n T_n - nk^2 r^{2n+3}\psi_{n+1}(kr)\frac{d}{dy}\frac{T_n}{r^{n+1}} \\ &\qquad + \psi_n(kr)\left(z\frac{d}{dx} - x\frac{d}{dz}\right) r^n U_n \\ w_n &= \frac{d}{dz} r^n S_n + (n+1)\,\psi_{n-1}(kr)\frac{d}{dz} r^n T_n - nk^2 r^{2n+3}\psi_{n+1}(kr)\frac{d}{dz}\frac{T_n}{r^{n+1}} \\ &\qquad + \psi_n(kr)\left(x\frac{d}{dy} - y\frac{d}{dx}\right) r^n U_n \end{aligned}\right\}. \tag{15}$$

The function $\psi_n(kr)$ is defined by (3), and S_n, T_n, U_n are spherical surface-harmonics of order n. The sign Σ indicates summation with respect to n, which may have the values 1, 2, 3, ... These formulæ make

$$xu + yv + zw = \Sigma\left\{nr^n S_n + n(n+1)(2n+1)\,\psi_n(kr)\,r^n T_n\right\}, \tag{16}$$

$$x\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + y\left(\frac{du}{dz} - \frac{dw}{dx}\right) + z\left(\frac{dv}{dx} - \frac{du}{dy}\right) = -\Sigma n(n+1)\,\psi_n(kr)\,r^n U_n. \tag{17}$$

When the extraneous disturbance consists of a train of plane waves, the harmonics S_n, T_n, U_n are determined as follows. Assuming that the direction of propagation is that of x-negative, and that the vibration is parallel to y, we may write, symbolically,

$$u = 0, \quad v = e^{ikx}, \quad w = 0, \tag{18}$$

the time-factor e^{ikct} being omitted, here and elsewhere, for shortness. If this be resolved into the sum of a series of disturbances of the ve $iky e^{ikz} = \Sigma\,(2n+1)(ikr)^n\,\psi_n\,(kr)\,\sin\theta\cos\omega P'_n\,(\cos\theta$

$P_n\,(\cos\theta)$ is the ordinary zonal harmonic. We infer
n with (19), that $S_n = 0$, and

$$T_n = \frac{(ik)^{n-1}}{n\,(n+1)}\sin\theta\cos\omega P'_n\,(\cos\theta).$$

rly, we find

$$U_n = -\frac{2n+1}{n\,(n+1)}\,(ik)^n\sin\theta\sin\omega P'_n\,(\cos\theta).$$

icular,

$$\sin\theta\cos\omega = \tfrac{1}{2}\frac{y}{r}, \qquad U_1 = -\tfrac{3}{2}ik\sin\theta\sin\omega =$$

$$k\sin\theta\cos\theta\cos\omega = \tfrac{1}{2}ik\frac{xy}{r^2}, \quad U_2 = \tfrac{5}{2}k^2\sin\theta\cos\theta\sin\omega$$

component tractions p_{rx}, p_{ry}, p_{rz} on a spherical su
described about the origin are given by

$$rp_{rx} = xp + \mu\left(r\frac{d}{dr} - 1\right)u + \mu\frac{d}{dx}(xu + yv + zw)\;\Bigg|$$

We have

$$\left.\begin{aligned} rF_n/\mu &= A_n(kr)\frac{d}{dx}r^nS_n+B_n(kr)\,r^{2n+1}\frac{d}{dx}\frac{S_n}{r^{n+1}}+C_n(kr)\frac{d}{dx}r^nT_n \\ &\quad +D_n(kr)\,r^{2n+1}\frac{d}{dx}\frac{T_n}{r^{n+1}}+E_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)r^nU_n \\ rG_n/\mu &= A_n(kr)\frac{d}{dy}r^nS_n+B_n(kr)\,r^{2n+1}\frac{d}{dy}\frac{S_n}{r^{n+1}}+C_n(kr)\frac{d}{dy}r^nT_n \\ &\quad +D_n(kr)\,r^{2n+1}\frac{d}{dy}\frac{T_n}{r^{n+1}}+E_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)r^nU_n \\ rH_n/\mu &= A_n(kr)\frac{d}{dz}r^nS_n+B_n(kr)\,r^{2n+1}\frac{d}{dz}\frac{S_n}{r^{n+1}}+C_n(kr)\frac{d}{dz}r^nT_n \\ &\quad +D_n(kr)\,r^{2n+1}\frac{d}{dz}\frac{T_n}{r^{n+1}}+E_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)r^nU_n \end{aligned}\right\}, \quad (28)$$

where
$$\left.\begin{aligned} A_n(kr) &= -\frac{k^2r^2}{2n+1}+2(n-1) \\ B_n(kr) &= \frac{k^2r^2}{2n+1} \end{aligned}\right\}, \qquad (29)$$

$$\left.\begin{aligned} C_n(kr) &= (n+1)\left\{kr\psi_{n-1}'(kr)+2(n-1)\,\psi_{n-1}(kr)\right\} \\ D_n(kr) &= -nk^2r^2\left\{kr\psi_{n+1}'(kr)-\psi_{n+1}(kr)\right\} \end{aligned}\right\}, \qquad (30)$$

$$E_n(kr) = kr\psi_n'(kr)+(n-1)\,\psi_n(kr).^* \qquad (31)$$

In the case of plane waves the terms in S_n, as we have seen, are absent.

Divergent Waves.

2. We have next to consider the expressions for the divergent waves. For these we have

$$\left.\begin{aligned} p &= -\mu k^2\Sigma\frac{S_n'}{r^{n+1}} \\ u = \Sigma u_n',\quad v &= \Sigma v_n',\quad w = \Sigma w_n' \end{aligned}\right\}, \qquad (32)$$

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[*] The steps of the calculation are indicated in Hydrodynamics, § 306.

where

$$\left.\begin{aligned} u'_n &= \frac{d}{dx}\frac{S'_n}{r^{n+1}} + (n+1) f_{n-1}(kr) \frac{d}{dx} r^n T'_n - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dx}\frac{T'_n}{r^{n+1}} \\ &\qquad + f_n(kr) \left(y\frac{d}{dz} - z\frac{d}{dy} \right) r^n U'_n \\ v'_n &= \frac{d}{dy}\frac{S'_n}{r^{n+1}} + (n+1) f_{n-1}(kr) \frac{d}{dy} r^n T'_n - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dy}\frac{T'_n}{r^{n+1}} \\ &\qquad + f_n(kr) \left(z\frac{d}{dx} - x\frac{d}{dz} \right) r^n U'_n \\ w'_n &= \frac{d}{dz}\frac{S'_n}{r^{n+1}} + (n+1) f_{n-1}(kr) \frac{d}{dz} r^n T'_n - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dz}\frac{T'_n}{r^{n+1}} \\ &\qquad + f_n(kr) \left(x\frac{d}{dy} - y\frac{d}{dx} \right) r^n U'_n \end{aligned}\right\}, \tag{33}$$

$f_n(kr)$ being the function defined by (5). The surface-harmonics S'_n, T'_n, U'_n are to be found in terms of S_n, T_n, U_n by means of the conditions which hold at the surface $r = a$. The admissible values of n are 1, 2, 3, ..., as before, the case $n = 0$ being excluded, since the volume enclosed by the spherical boundary will be assumed to be constant. These formulæ make

$$xu + yv + zw = \Sigma \left\{ -(n+1)\frac{S'_n}{r^{n+1}} + n(n+1)(2n+1) f_n(kr)\, r^n T'_n \right\}, \tag{34}$$

$$x\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + y\left(\frac{du}{dz} - \frac{dw}{dx}\right) + z\left(\frac{dv}{dx} - \frac{du}{dy}\right) = -\Sigma n(n+1) f_n(kr)\, T'_n. \tag{35}$$

Again, if we denote the parts of p_{rx}, p_{ry}, p_{rz} which are due to (u'_n, v'_n, w'_n) by F'_n, G'_n, H'_n, we find

$$\left.\begin{aligned} rF'_n/\mu &= A'_n(kr)\frac{1}{r^{2n+1}}\frac{d}{dx} r^n S'_n + B'_n(kr)\frac{d}{dx}\frac{S'_n}{r^{n+1}} + C'_n(kr)\frac{d}{dx} r^n T'_n \\ &\qquad + D'_n(kr)\, r^{2n+1}\frac{d}{dx}\frac{T'_n}{r^{n+1}} + E'_n(kr)\left(y\frac{d}{dz} - z\frac{d}{dy}\right) r^n U_n \\ rG'_n/\mu &= A'_n(kr)\frac{1}{r^{2n+1}}\frac{d}{dy} r^n S'_n + B'_n(kr)\frac{d}{dy}\frac{S'_n}{r^{n+1}} + C'_n(kr)\frac{d}{dy} r^n T_n \\ &\qquad + D'_n(kr)\, r^{2n+1}\frac{d}{dy}\frac{T'_n}{r^{n+1}} + E'_n(kr)\left(z\frac{d}{dx} - x\frac{d}{dz}\right) r^n U'_n \\ rH'_n/\mu &= A'_n(kr)\frac{1}{r^{2n+1}}\frac{d}{dz} r^n S'_n + B'_n(kr)\frac{d}{dz}\frac{S'_n}{r^{n+1}} + C'_n(kr)\frac{d}{dz} r^n T'_n \\ &\qquad + D'_n(kr)\, r^{2n+1}\frac{d}{dz}\frac{T'_n}{r^{n+1}} + E'_n(kr)\left(x\frac{d}{dy} - y\frac{d}{dx}\right) r^n U'_n \end{aligned}\right\}, \tag{36}$$

where
$$\left.\begin{aligned} A'_n(kr) &= -\frac{k^2r^2}{2n+1} \\ B'_n(kr) &= \frac{k^2r^2}{2n+1} - 2(n+2) \end{aligned}\right\}, \tag{37}$$

$$\left.\begin{aligned} C'_n(kr) &= (n+1)\left\{kr f'_{n-1}(kr) + 2(n-1) f_{n-1}(kr)\right\} \\ D'_n(kr) &= -nk^2r^2\left\{kr f'_{n+1}(kr) - f_{n+1}(kr)\right\} \end{aligned}\right\}, \tag{38}$$

$$E'_n(kr) = kr f'_n(kr) + (n-1) f_n(kr)\}. \tag{39}$$

3. We have to examine the properties of the scattered waves at a great distance from the origin. When kr is large, we have

$$f_n(kr) = \frac{i^n}{(kr)^{n+1}} e^{-ikr}, \tag{40}$$

approximately, and therefore, keeping only the most important terms,

$$\left.\begin{aligned} u'_n &= (2n+1)\frac{i^{n-1}}{(kr)^n} e^{-ikr}\left(\frac{d}{dx} r^n T'_n - nxr^{n-2} T'_n\right) \\ &\qquad + \frac{i^n}{(kr)^{n+1}} e^{-ikr}\left(y\frac{d}{dz} - z\frac{d}{dy}\right) r^n U'_n \\ v'_n &= (2n+1)\frac{i^{n-1}}{(kr)^n} e^{-ikr}\left(\frac{d}{dy} r^n T'_n - nyr^{n-2} T'_n\right) \\ &\qquad + \frac{i^n}{(kr)^{n+1}} e^{-ikr}\left(z\frac{d}{dx} - x\frac{d}{dz}\right) r^n U'_n \\ w'_n &= (2n+1)\frac{i^{n-1}}{(kr)^n} e^{-ikr}\left(\frac{d}{dz} r^n T'_n - nzr^{n-2} T'_n\right) \\ &\qquad + \frac{i^n}{(kr)^{n+1}} e^{-ikr}\left(x\frac{d}{dy} - y\frac{d}{dx}\right) r^n U'_n \end{aligned}\right\}, \tag{41}$$

where a slight transformation has been effected by means of the formula (8). We notice that u'_n, v'_n, w'_n are ultimately of the order $1/r$, whilst the radial displacement $(xu'_n + yv'_n + zw'_n)/r$ is zero to the present degree of approximation. It is really of the order $1/r^2$, as is easily seen from (34).

The geometrical interpretation of the two types of terms which appear in (41) is very simple. The terms in T'_n express a state of motion in which the particles on a sphere of large radius r are vibrating in directions orthogonal to the contour-lines of the harmonic T'_n, with an amplitude inversely proportional to the distance between consecutive contours, if we imagine these to be drawn for equal infinitesimal increments of T'_n. The terms in U'_n, on the other hand, express a vibration *along* the contour lines of U'_n, with a similar law of amplitude.*

For example, if (in accordance with a later notation) we write

$$rT'_1 = \tfrac{1}{2}B'_1 y, \quad rU'_1 = 0, \tag{42}$$

we have
$$(u'_1,\ v'_1,\ w'_1) = \frac{3B'_1}{2k}\left(-\frac{xy}{r^2},\ \frac{x^2+z^2}{r^2},\ -\frac{yz}{r^2}\right)\frac{e^{-ikr}}{r}. \tag{43}$$

The particles on a sphere of large radius r are here vibrating in the directions of the meridians in which the sphere is cut by planes through the axis of y, with an amplitude proportional to the distance from this axis.

On the other hand, if

$$rT'_1 = 0, \quad rU'_1 = -\tfrac{3}{2}ikC'_1 z, \tag{44}$$

we find
$$(u'_1,\ v'_1,\ w'_1) = \frac{3C'_1}{2k}\left(\frac{y}{r},\ -\frac{x}{r},\ 0\right)\frac{e^{-ikr}}{r}. \tag{45}$$

The lines of motion are circles about the axis of z, and the amplitude, for points at the same distance r from the origin, varies as the distance from this axis.

Again, if
$$r^2 T'_2 = \tfrac{1}{2}ikB'_2 xy, \quad U'_2 = 0, \tag{46}$$

we find

$$(u'_2,\ v'_2,\ w'_2) = -\frac{5B'_2}{2k}\left(\frac{y(y^2+z^2-x^2)}{r^3},\ \frac{x(z^2+x^2-y^2)}{r^3},\ \frac{2xyz}{r^3}\right)\frac{e^{-ikr}}{r}. \tag{47}$$

The lines of motion on a sphere of radius r are the orthogonal trajectories of the curves in which the sphere is cut by the hyperbolic cylinders $xy = \text{const}$. It is easily proved that these trajectories are the intersections of the sphere with the cones $z^2 = \lambda(x^2-y^2)$, where λ is a variable parameter. They all pass through the four points

$$\left(\pm\frac{r}{\sqrt{2}},\ \pm\frac{r}{\sqrt{2}},\ 0\right).$$

The values of F'_n, G'_n, H'_n at a great distance from the origin are most easily obtained from (27) and (41). Having regard to the order of magnitude of u'_n, v'_n, w'_n and $(xu'_n+yv'_n+zw'_n)/r$, we readily find

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* *Cf. Proc. Lond. Math. Soc.*, Vol. XIII., p. 194.

$$F'_n = -ik\mu u'_n, \quad G'_n = -ik\mu v'_n, \quad H'_n = -ik\mu w'_n. \tag{48}$$

It appears that the radial stress is relatively very small, as might have been anticipated from the fact that the waves tend to become ultimately plane.

Propagation of Energy Outwards.

4. The formulæ (41) and (48) enable us to calculate a very useful expression for the rate at which energy is carried off by the system of scattered waves represented by (32), (33). If, after restoring the time-factor, we take only the real parts of our expressions, the rate in question is equal to the mean value (with respect to the time) of the integral

$$-\iint\left(p_{rx}\frac{du}{dt}+p_{ry}\frac{dv}{dt}+p_{rz}\frac{dw}{dt}\right)dS, \tag{49}$$

taken over the surface of a sphere, of *any* radius r, described about the origin. For simplicity of calculation we may suppose r to be infinitely great.

It appears on examination that, if m and n be distinct,

$$\iint\left(F'_m\frac{du'_n}{dt}+G'_m\frac{dv'_n}{dt}+H'_m\frac{dw'_n}{dt}\right)dS=0, \tag{50}$$

in virtue of the conjugate property of spherical harmonics of different orders, so that the expression (49) reduces to the form

$$-\Sigma\iint\left(F'_n\frac{du'_n}{dt}+G'_n\frac{dv'_n}{dt}+H'_n\frac{dw'_n}{dt}\right)dS. \tag{51}$$

Moreover, it is easily seen that the terms in this expression which involve combinations of T'_n with U'_n will disappear.*

Let us now write in (41)

$$r^n T'_n = \Phi_n + i\phi_n. \tag{52}$$

Taking real parts, we have, so far as the terms in T'_n are concerned,

$$\frac{du'_n}{dt} = kc\frac{2n+1}{(kr)^n}\left\{\left(\frac{d\Phi_n}{dx}-n\frac{x}{r^2}\Phi_n\right)\cos k(ct-r+\epsilon)\right.$$
$$\left.-\left(\frac{d\phi_n}{dx}-n\frac{x}{r^2}\phi_n\right)\sin k(ct-r+\epsilon)\right\}, \tag{53}$$

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* See *Camb. Trans.*, Vol. xviii., p. 357.

where ϵ may be 0 or π or $\pm\frac{1}{2}\pi$. Also

$$-F'_n = \mu k \frac{2n+1}{(kr)^n} \{\ldots\}, \tag{54}$$

where the terms in { } are the same as in (53). Hence, so far as it depends on T'_n, the mean value of the expression (51) will be

$$\tfrac{1}{2}\mu k^2 c\,\frac{(2n+1)^2}{(kr)^{2n}} \iint \left\{ \left(\frac{d\Phi_n}{dx} - n\frac{x}{r^2}\Phi_n\right)^2 + \left(\frac{d\Phi_n}{dy} - n\frac{y}{r^2}\Phi_n\right)^2 + \left(\frac{d\Phi_n}{dz} - n\frac{z}{r^2}\Phi_n\right)^2 \right.$$

$$\left. + \left(\frac{d\phi_n}{dx} - n\frac{x}{r^2}\phi_n\right)^2 + \left(\frac{d\phi_n}{dy} - n\frac{y}{r^2}\phi_n\right)^2 + \left(\frac{d\phi_n}{dz} - n\frac{z}{r^2}\phi_n\right)^2 \right\} d$$

$$= \tfrac{1}{2}\mu k^2 c\,\frac{(2n+1)^2}{(kr)^{2n}} \iint \left\{ \left(\frac{d\Phi_n}{dx}\right)^2 + \left(\frac{d\Phi_n}{dy}\right)^2 + \left(\frac{d\Phi_n}{dz}\right)^2 - \frac{n^2}{r^2}\Phi_n^2 \right.$$

$$\left. + \left(\frac{d\phi_n}{dx}\right)^2 + \left(\frac{d\phi_n}{dy}\right)^2 + \left(\frac{d\phi_n}{dz}\right)^2 - \frac{n^2}{r^2}\phi_n^2 \right\} dS. \qquad (55$$

In the same way, if

$$r^n U'_n = \mathrm{X}_n + i\chi_n, \tag{56}$$

we find that the mean value of (51), so far as it depends on U'_n, will be

$$\tfrac{1}{2}\frac{\mu k^3 c}{(kr)^{2n+2}} \iint \left\{ \left(y\frac{d\mathrm{X}_n}{dz} - z\frac{d\mathrm{X}_n}{dy}\right)^2 + \left(z\frac{d\mathrm{X}_n}{dx} - x\frac{d\mathrm{X}_n}{dz}\right)^2 + \left(x\frac{d\mathrm{X}_n}{dy} - y\frac{d\mathrm{X}_n}{dx}\right)^2 \right.$$

$$\left. + \left(y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right)^2 + \left(z\frac{d\chi_n}{dx} - x\frac{d\chi_n}{dz}\right)^2 + \left(x\frac{d\chi_n}{dy} - y\frac{d\chi_n}{dx}\right)^2 \right\} dS$$

$$= \tfrac{1}{2}\frac{\mu c}{(kr)^{2n}} \iint \left\{ \left(\frac{d\mathrm{X}_n}{dx}\right)^2 + \left(\frac{d\mathrm{X}_n}{dy}\right)^2 + \left(\frac{d\mathrm{X}_n}{dz}\right)^2 - \frac{n^2}{r^2}\mathrm{X}_n^2 \right.$$

$$\left. + \left(\frac{d\chi_n}{dx}\right)^2 + \left(\frac{d\chi_n}{dy}\right)^2 + \left(\frac{d\chi_n}{dz}\right)^2 - \frac{n^2}{r^2}\chi_n^2 \right\} dS. \qquad (5$$

The expressions under the integral signs in (55) and (57) are the sums of the squares of the tangential components of the vectors $(d\Phi_n/dx,\ d\Phi_n/dy,\ d\Phi_n/dz)$, &c., as might have been foreseen from the geometrical interpretation indicated in § 3. Now, turning to polar coordinates, if Ω_n be a surface-harmonic of order n, we have

$$\int_0^{2\pi}\!\!\int_0^{\pi} \left\{ \left(\frac{d\Omega_n}{d\theta}\right)^2 + \left(\frac{d\Omega_n}{\sin\theta\, d\omega}\right)^2 \right\} \sin\theta\, d\theta\, d\omega = n(n+1)\int_0^{2\pi}\!\!\int_0^{\pi} \Omega_n^2 \sin\theta\, d\theta\, d\omega. \tag{58}$$

Hence, summing the expressions (55) and (57), we obtain

$$\tfrac{1}{2}\mu c\,\Sigma\frac{n(n+1)}{k^{2n}}\int_0^{2\pi}\!\!\int_0^{\pi}\left\{(2n+1)^2k^2\,|\,T'_n\,|^2+|\,U'_n\,|^2\right\}\sin\theta\,d\theta\,d\omega. \tag{59}$$

In the case of incident plane waves, the formulæ (23) and (24) suggest that we should write

$$T'_n = B'_n T_n = B'_n\frac{(ik)^{n-1}}{n(n+1)}\sin\theta\cos\omega\,P'_n(\cos\theta), \tag{60}$$

$$U'_n = C'_n U_n = -C'_n\frac{2n+1}{n(n+1)}(ik)^n\sin\theta\sin\omega\,P'_n(\cos\theta). \tag{61}$$

Since $\displaystyle\int_0^{2\pi}\!\!\int_0^{\pi}\left\{\sin\theta\cos\omega\,P'_n(\cos\theta)\right\}^2\sin\theta\,d\theta\,d\omega = \frac{2\pi n(n+1)}{2n+1}$, (62)

the expression (59) now takes the form

$$\pi\mu c\,\Sigma(2n+1)\left\{|\,B'_n\,|^2+|\,C'_n\,|^2\right\}. \tag{63}$$

The proper standard of comparison here is the energy propagated per unit time across unit area (of the wave-front) in the primary waves represented by (18). This is equal to $\frac{1}{2}\mu k^2 c$. Hence, denoting by I the ratio which the energy dissipated by the scattered waves bears to the energy-flux in the primary waves, we have

$$I = \frac{2\pi}{k^2}\Sigma(2n+1)\left\{|\,B'_n\,|^2+|\,C'_n\,|^2\right\}. \tag{64}$$

This has exactly the same form as in the corresponding electromagnetic problem.*

Fixed Rigid Obstacle.

5. We may first suppose that the internal boundary ($r = a$) is absolutely fixed. The surface-conditions then are

$$\Sigma(u_n+u'_n) = 0,\quad \Sigma(v_n+v'_n) = 0,\quad \Sigma(w_n+w'_n) = 0. \tag{65}$$

If we substitute the values of u_n, v_n, w_n, u'_n, v'_n, w'_n from (15) and (33), we deduce†

$$S_n+(n+1)\,\psi_{n-1}(ka)\,T_n+(n+1)\,f_{n-1}(ka)\,T'_n = 0, \tag{66}$$

$$-nk^2a^2\psi_{n+1}(ka)\,T_n+\frac{S'_n}{a^{2n+1}}-nk^2a^2f_{n+1}(ka)\,T'_n = 0, \tag{67}$$

$$\psi_n(ka)\,U_n+f_n(ka)\,U'_n = 0. \tag{68}$$

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* *Camb. Trans.*, Vol. xviii., p. 358. [See also Lorenz, *loc. cit.*, p. 492.]

† The proof of these and similar inferences in the sequel is indicated in *Hydrodynamics*, § 307.

These determine the harmonics $S_n^{\backprime}$, $T_n^{\backprime}$, $U_n^{\backprime}$ in terms of S_n, T_n, U_n, which are regarded as given.

When there is no extraneous disturbance, we have $S_n = 0$, $T_n = 0$, $U_n = 0$. The free vibrations involving $T_n^{\backprime}$ have values of k determined by

$$f_{n-1}(ka) = 0\,; \tag{69}$$

thus, for $n = 2$, we find $ika = -1$. Since the time-factor is e^{ikct}, this gives an "aperiodic" vibration $e^{-ct/a}$. The free vibrations involving $U_n^{\backprime}$ have values of k determined by

$$f_n(ka) = 0. \tag{70}$$

The characteristic of all these free vibrations, whether oscillatory or not, is that the "modulus of decay" is comparable with the time a plane wave would take to traverse a distance equal to the radius of the sphere.

In the case of plane incident waves, we have $S_n = 0$; and, if we further suppose that the wave-length is large compared with the dimensions of the spherical obstacle, ka will be a small quantity. Hence, keeping only the most important terms, we find

$$T_n^{\backprime} = -\frac{\psi_{n-1}(ka)}{f_{n-1}(ka)}\,T_n = -\frac{(ka)^{2n-1}}{\{1.3\ldots(2n-3)\}^2(2n-1)}\,T_n, \tag{71}$$

$$U_n = -\frac{\psi_n(ka)}{f_n(ka)}\,U_n = -\frac{(ka)^{2n+1}}{\{1.3\ldots(2n-1)\}^2(2n+1)}\,U_n. \tag{72}$$

The most important part of the scattered waves is that represented by $T_1^{\backprime}$. Since, from (71) and (25),

$$T_1^{\backprime} = -kaT'_1 = -\tfrac{1}{2}ka\,\frac{y}{r}\,, \tag{73}$$

the corresponding vibration $(u_1^{\backprime}, v_1^{\backprime}, w_1^{\backprime})$ at a distance from the origin is found by making $B_1^{\backprime} = -ka$ in (43). This type of vibration is familiar in problems such as we are engaged on, but the amplitude is now much greater than usual, viz., in a ratio of the order $(ka)^{-2}$. This is accounted for by the abnormal degree of constraint imposed on our medium at a rigid and immovable boundary. In the corresponding acoustical problem,* where plane waves of sound impinge on a fixed sphere, there is complete freedom of lateral motion at the surface.

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* Lord Rayleigh, *Theory of Sound*, § 334.

The formula (64) gives for the dissipation-ratio in the present case

$$I = 6\pi a^2, \tag{74}$$

approximately.

Rigid Movable Sphere.

6. Next suppose that a perfectly rigid but movable sphere is embedded in the medium. The tractions on the surface will reduce to a force (X, Y, Z) and a couple (L, M, N). When the incident wave-system is that given by (18), the components Y of force and N of couple will (in the absence of gyrostatic influence) be the only ones which do not vanish, and the motion of the sphere will consist of a linear oscillation in the axis of y and an angular oscillation about the axis of z. It is convenient, however, in the first instance, to treat the problem more generally. In any case it appears that the only parts of our general formulæ which will contribute anything to the values of X, Y, Z, L, M, N are those for which $n = 1$; moreover, that the linear and angular motions of the sphere are independent, the former depending on the harmonics S_1, T_1, and the latter on U_1. For $n > 1$, the determination of S_n', T_n', U_n' in terms of S_n, T_n, U_n will be exactly the same as when the sphere was fixed (§ 5).

We find

$$\left.\begin{aligned} X &= \iint (F_1 + F_1')\, dS \\ Y &= \iint (G_1 + G_1')\, dS \\ Z &= \iint (H_1 + H_1')\, dS \end{aligned}\right\}, \tag{75}$$

$$\left.\begin{aligned} L &= \iint \{y(H_1 + H_1') - z(G_1 + G_1')\}\, dS \\ M &= \iint \{z(F_1 + F_1') - x(H_1 + H_1')\}\, dS \\ N &= \iint \{x(G_1 + G_1') - y(F_1 + F_1')\}\, dS \end{aligned}\right\}, \tag{76}$$

where the integrations extend over the surface $r = a$. Hence, substituting from (28) and (36), we find

$$\begin{aligned} X = -\tfrac{4}{3}\pi\mu k^2 a^3 &\left\{ \frac{d}{dx} rS_1 + \frac{1}{a^3}\, \frac{d}{dx} rS_1' \right\} \\ &+ 8\pi\mu ka^2 \left\{ \psi_0'(ka) \frac{d}{dx} rT_1 + f_0'(ka) \frac{d}{dx} rT_1' \right\}, \end{aligned} \tag{77}$$

$$L = -\tfrac{8}{3}\pi\mu ka^4 \left\{ \psi_1'(ka)\, \frac{d}{dx} rU_1 + f_1'(ka\ \frac{d}{dx} rU_1' \right\}, \tag{78}$$

with similar equations. We may therefore write

$$-\tfrac{4}{3}\pi\mu k^2a^4\left(S_1+\frac{S_1'}{a^3}\right)+8\pi\mu ka^3\left\{\psi_0'(ka)\,T_1+f_0'(ka)\,T_1'\right\}=Xx+Yy+Zz, \tag{79}$$

$$-\tfrac{8}{3}\pi\mu ka^5\left\{\psi_1'(ka)\,U_1+f_1'(ka)\,U_1'\right\}=Lx+My+Nz. \tag{80}$$

Again, if (α, β, γ) and (ξ, η, ζ) be the linear and angular displacements of the sphere, the kinematical relations to be satisfied at the surface $r = a$ are

$$u_1+u_1'=\alpha+\eta z-\zeta y,\quad v_1+v_1'=\beta+\zeta x-\xi z,\quad w_1+w_1'=\gamma+\xi y-\eta x. \tag{81}$$

Hence, by (16), (17), and (34), (35),

$$a\left(S_1-2\frac{S_1'}{a^3}\right)+6a\left\{\psi_1(ka)\,T_1+f_1(ka)\,T_1'\right\}=\alpha x+\beta y+\gamma z, \tag{82}$$

$$-a\left\{\psi_1(ka)\,U_1+f_1(ka)\,U_1'\right\}=\xi x+\eta y+\zeta z. \tag{83}$$

It follows also from (15) and (33) that we must have

$$\frac{S_1'}{a^3}-k^2a^2\left\{\psi_2(ka)\,T_1+f_2(ka)\,T_1'\right\}=0, \tag{84}$$

since the surface-harmonics of order 2, in the values of u_1+u_1', v_1+v_1', w_1+w_1' must cancel.

We have, further, the dynamical equations

$$m\frac{d^2\alpha}{dt^2}=X,\quad m\frac{d^2\beta}{dt^2}=Y,\quad m\frac{d^2\gamma}{dt^2}=Z, \tag{85}$$

$$K\frac{d^2\xi}{dt^2}=L,\quad K\frac{d^2\eta}{dt^2}=M,\quad K\frac{d^2\zeta}{dt^2}=N, \tag{86}$$

where m denotes the mass of the sphere, and K its moment of inertia. There is an obvious convenience in writing

$$m=\kappa\,.\,\tfrac{4}{3}\pi\rho a^3,\quad K=\nu\,.\,\tfrac{4}{3}\pi\rho a^3\,.\,\tfrac{2}{5}a^2. \tag{87}$$

Remembering that the time-factor is e^{ikct}, and that $c = \mu/\rho$, the equations (85), (86) give

$$Xx+Yy+Zz=-\kappa\,.\,\tfrac{4}{3}\pi\mu k^2a^3(\alpha x+\beta y+\gamma z), \tag{88}$$

$$Lx+My+Nz=-\nu\,.\,\tfrac{8}{15}\pi\mu k^2a^5(\xi x+\eta y+\zeta z). \tag{89}$$

Hence, from (79) and (82),

$$(\kappa-1)\,S_1-(2\kappa+1)\frac{S_1'}{a^3}+6(\kappa-1)\left\{\psi_1(ka)\,T_1+f_1(ka)\,T_1'\right\}=0. \tag{90}$$

Combined with (84) this gives

$$(\kappa-1)\,S_1+\left\{(2\kappa+1)\,\psi_0(ka)-9\psi_1(ka)\right\}T_1$$
$$+\left\{(2\kappa+1)\,f_0\,(ka)-9f_1\,(ka)\right\}T_1' = 0, \quad (91)$$

where a simplification has been made by means of (6) and (7).

Similarly, from (80), (83), and (89), we deduce

$$\left\{\tfrac{1}{5}\nu ka\,\psi_1(ka)+\psi_1'(ka)\right\}U_1+\left\{\tfrac{1}{5}\nu ka\,f_1\,(kx)+f_1'\,(ka)\right\}U_1' = 0. \quad (92)$$

The free *rectilinear* vibrations which the sphere can execute in the absence of extraneous disturbance are found by putting $S_1 = 0$, $T_1 = 0$ in (91), whence

$$(2\kappa+1)f_0(ka)-9f_1(ka) = 0, \quad (93)$$

or

$$(2\kappa+1)(ika)^2+9\,(ika)+9 = 0. \quad (94)$$

If $\kappa > \tfrac{5}{8}$, the roots of this quadratic in *ika* are complex, with a real negative part, viz.,

$$ika = -\frac{9}{4\kappa+2}\pm i\,\frac{3\sqrt{(8\kappa-5)}}{4\kappa+2}. \quad (95)$$

This indicates a periodic oscillation with a gradually diminishing amplitude. When the ratio (κ) which the mass of the sphere bears to that of an equal volume of the medium is a large number, the wave-length corresponding to the period will be large compared with the radius of the sphere, and a considerable number of oscillations will take place in the interval required for the amplitude to diminish in the ratio $1/e$. When $\kappa < \tfrac{5}{8}$, the values of *ika* are real and negative, and the vibration is aperiodic.

Again, the free *rotational* vibrations of the sphere are found by putting $U_1 = 0$ in (92); thus

$$\tfrac{1}{5}\nu ka\,f_1\,(ka)+f_1'\,(ka) = 0, \quad (96)$$

or

$$\nu\,(ika)^3+(\nu+5)(ika)^2+15ika+15 = 0. \quad (97)$$

When ν is large one root of this cubic in *ika* is $-1-\tfrac{1}{5}\nu$, nearly, corresponding to an aperiodic motion. The others are given by

$$ika = -\frac{75}{2\nu^2}\pm i\sqrt{\left(\frac{15}{\nu}\right)}, \quad (98)$$

approximately; and it is to be observed that the modulus of decay is, relatively to the period, very much longer than in the case of the rectilinear vibrations.

In the case of the plane incident system (18) we have $S_1 = 0$, and the values of T'_1 and U'_1 in terms of T_1 and U_1, respectively, are given by (91) and (92). When κ and ν have only moderate values, simple approximate expressions are easily found for the case where ka is small; thus

$$T'_1 = \tfrac{2}{9}(\kappa-1)\,k^3a^3T_1, \quad U'_1 = \tfrac{1}{45}(\nu-1)\,k^5a^5U_1, \tag{99}$$

showing that the *angular* motion of the sphere is of secondary importance. We must remember that T'_2 is now of equal rank with T'_1, being given by (71). The most important parts of the disturbance at a distance are therefore found by putting

$$B'_1 = \tfrac{2}{9}(\kappa-1)\,k^3a^3, \quad B'_2 = -\tfrac{1}{3}k^3a^3, \tag{100}$$

in (43) and (47) respectively. The dissipation-ratio is

$$I = \left\{\tfrac{8}{27}(\kappa-1)^2+\tfrac{40}{81}\right\}\pi a^2(ka)^4, \tag{101}$$

approximately, showing the usual law of dependence upon wave-length.

The expressions (99) vanish, to our present degree of approximation, when $\kappa = 1$, $\nu = 1$. For $\kappa = 0$, $\nu = 0$ they coincide with the results obtained below in § 9 for the case of a spherical cavity in the medium.

When, on the other hand, κ and ν are so large as to be comparable with $(ka)^{-2}$, the above approximation is insufficient. It is clear however that, except in the neighbourhood of the critical wave-lengths referred to in § 11, the results must tend to coincide with those obtained in § 5 for a fixed sphere, and in particular the dissipation-ratio must tend to the value (74).

Rigid Shell containing a Vibratory Mechanism.

7. The investigation of the preceding section may be readily extended in various ways, by imagining that the sphere is hollow and contains a dynamical system capable of storing potential and kinetic energy. Various arrangements of this kind have been suggested by Lord Kelvin. As a sufficient example of a system devoid of gyrostatic influence, we may take the case of a rigid spherical shell containing a central spherical nucleus attached to it by springs.

To calculate first the linear vibration parallel to y, such as will be called into play by waves of the type (18), let m denote the mass of the shell, m_0 that of the nucleus, and let β, β_0 be the displacements of their centres. If $2\pi/\sigma_0$ be the period of vibration of the nucleus when the shell is held fixed, we have

$$\frac{d^2\beta_0}{dt^2} = -\sigma_0^2(\beta_0-\beta) \tag{102}$$

and

$$m\frac{d^2\beta}{dt^2} + m_0\frac{d^2\beta_0}{dt^2} = Y, \tag{103}$$

where Y denotes, as before, the effect of the stresses on the outer surface. Writing $-k^2c^2$ for d^2/dt^2, and eliminating β_0, we find

$$-\frac{(m+m_0)\sigma_0^2 - mk^2c^2}{\sigma_0^2-k^2c^2}\,k^2c^2\beta = Y. \tag{104}$$

If we put

$$m = \kappa\,.\,\tfrac{4}{3}\pi\rho a^3, \quad m_0 = \kappa_0\,.\,\tfrac{4}{3}\pi\rho a^3, \tag{105}$$

it appears that to adapt the various results of § 6 to the present circumstances we have merely to replace κ by

$$\kappa' = \frac{(\kappa+\kappa_0)\sigma_0^2 - \kappa k^2c^2}{\sigma_0^2-k^2c^2}. \tag{106}$$

The limiting form of the result when the inertia of the nucleus is very great is obtained by putting $\kappa_0 = \infty$, $\sigma_0 = 0$, whilst $\kappa_0\sigma_0^2$ is finite. The shell then behaves as if (in addition to the force Y) it were urged to a fixed position by a force varying as the distance.

Similarly, if the springs resist rotation of the nucleus relatively to the containing shell, we have to replace ν by

$$\nu' = \frac{(\nu+\nu_0)\sigma_0^2 - \nu k^2c^2}{\sigma_0^2-k^2c^2}, \tag{107}$$

where the constants ν, ν_0, σ_0 have analogous meanings, as regards rotation, to those adopted for κ, κ_0, σ_0 in the case of translation.

It is evident in each case that the approximations we have made on the assumption that ka is small will require revision when ka is nearly equal to $\sigma_0 a/c$. The further consideration of such questions is, however, deferred to § 11.

Rigid Shell containing a Gyrostat.

8. To examine the kind of modification that will be introduced into our results by gyrostatic influence, we may imagine that the sphere of § 6 is replaced by a spherical shell containing a gyrostat in rapid rotation. The first part of the investigation proceeds as before,

until we come to the dynamical equations of rotation (86). These now take the forms

$$\left.\begin{aligned} K\frac{d^2\xi}{dt^2}+R\frac{d\eta}{dt}-Q\frac{d\zeta}{dt}&=L \\ K\frac{d^2\eta}{dt^2}+P\frac{d\zeta}{dt}-R\frac{d\xi}{dt}&=M \\ K\frac{d^2\zeta}{dt^2}+Q\frac{d\xi}{dt}-P\frac{d\eta}{dt}&=N \end{aligned}\right\} . \tag{108}$$

The symbols P, Q, R here denote the components of angular momentum of the gyrostat when the containing shell is at rest in its equilibrium position.

Special interest attaches to the two cases where the axis of the gyrostat is parallel and perpendicular, respectively, to the direction of propagation of the plane waves.

In the former case we have $Q = R = 0$, and the equations (108) reduce to

$$K\frac{d^2\xi}{dt^2}=L,\quad K\frac{d^2\eta}{dt^2}+P\frac{d\zeta}{dt}=M,\quad K\frac{d^2\zeta}{dt^2}-P\frac{d\eta}{dt}=N. \tag{109}$$

The last two are equivalent to

$$\left.\begin{aligned} K\frac{d^2}{dt^2}(\eta+i\zeta)-iP\frac{d}{dt}(\eta+i\zeta) &= M+iN \\ K\frac{d^2}{dt^2}(\eta-i\zeta)+iP\frac{d}{dt}(\eta-i\zeta) &= M-iN \end{aligned}\right\} . \tag{110}$$

The rotational oscillations about the axis of x call for no special treatment. In the remaining types we have $\xi = 0$, $L = 0$, and *either*

$$\left.\begin{aligned} \eta = i\zeta,\quad M &= iN \\ \text{with}\qquad K\frac{d^2\eta}{dt^2}-iP\frac{d\eta}{dt} &= M \end{aligned}\right\} , \tag{111}$$

or

$$\left.\begin{aligned} \eta = -i\zeta,\quad M &= -iN \\ \text{with}\qquad K\frac{d^2\eta}{dt^2}+iP\frac{d\eta}{dt} &= M \end{aligned}\right\} . \tag{112}$$

In the case of (111) we may write

$$rU_1 = B(y-iz),\quad rU_1' = B'(y-iz). \tag{113}$$

Hence, from (83) and (80),

$$\eta = -\{B\psi_1(ka) + B'f_1(ka)\}, \tag{114}$$

$$M = -\tfrac{8}{3}\pi\mu ka^4\{B\psi_1'(ka) + B'f_1'(ka)\}. \tag{115}$$

If we write $$K = \nu . \tfrac{4}{3}\pi\rho a^3 . \tfrac{2}{5}a^2, \quad P = K\varpi, \tag{116}$$

ϖ being a quantity of the nature of an angular velocity, and if we assume the time-factor e^{ikct} in (111), we find

$$\left\{\tfrac{1}{5}\nu\left(ka - \frac{\varpi a}{c}\right)\psi_1(ka) + \psi_1'(ka)\right\}B$$
$$+ \left\{\tfrac{1}{5}\nu\left(ka - \frac{\varpi a}{c}\right)f_1(ka) + f_1'(ka)\right\}B' = 0, \tag{117}$$

which is the altered form of (92).

In the *free* oscillations of the type (111) we have $B = 0$, and

$$\tfrac{1}{5}\nu\left(ka - \frac{\varpi a}{c}\right)f_1(ka) + f_1'(ka) = 0, \tag{118}$$

or $$\tfrac{1}{5}\nu ika(ika+1)\left(ika - i\frac{\varpi a}{c}\right) + (ika)^2 + 3ika + 3 = 0. \tag{119}$$

If ν is a large number, one root of this is $ika = -1$, nearly; this indicates a rapidly decaying vibration. If, further, $\varpi a/c$ be small compared with $\sqrt{(15/\nu)}$, the remaining roots will be

$$ika = ik_1a, \quad -ik_2a, \tag{120}$$

where $$k_1a = \sqrt{\frac{15}{\nu}} + \tfrac{1}{2}\frac{\varpi a}{c}, \quad k_2a = \sqrt{\frac{15}{\nu}} - \tfrac{1}{2}\frac{\varpi a}{c}, \tag{121}$$

nearly. It may be shown that in the motion corresponding to the positive value of k the axis of the gyrostat has a positive precessional motion about the axis of x, whilst the negative root gives a negative precessional motion. The period ($2\pi/k_1c$ or $2\pi/k_2c$) is shorter when the direction of the precessional motion agrees with that of the rotation of the gyrostat than when it is opposed to it.

If we substitute from (113) in (41), we find that at a distance from the sphere

$$(u_1', v_1', w_1') = \frac{iB'}{k^2r}\left(-\frac{iy+z}{r}, \frac{ix}{r}, \frac{x}{r}\right)\begin{cases} e^{ik_1(ct-r)} \\ e^{-ik_2(ct-r} \end{cases}. \tag{122}$$

At points on the axis of x the vibrations are circular; in the wave corresponding to k_1 they have the same sense as ϖ, and in the wave corresponding to k_2 they have the sense opposite to that of ϖ. At points of the plane yz the vibrations are rectilinear, and parallel to x.

The consideration of the system (112) leads to similar conclusions.

To examine the effect of the obstacle on a system of plane waves it is necessary to analyse these into right-handed and left-handed circular vibrations. For a train of right-handed circular vibrations, say

$$u = 0, \quad v = e^{ikx}, \quad w = -ie^{ikx}, \tag{123}$$

the formulæ (113) will apply, provided $B = \frac{3}{2}k$. The value of B' is then given by (117), and the scattered waves by

$$(u_1', v_1', w_1') = \frac{iB'}{k^2 r}\left(-\frac{iy+z}{r}, \frac{ix}{r}, \frac{x}{r}\right) e^{ik(ct-r)}, \tag{124}$$

which has just been interpreted. The case of a train of left-handed vibrations could be treated in a similar manner, but the result will evidently be equivalent to reversing the sign of ϖ in (117).

To investigate the case where the axis of the gyrostat is perpendicular to the direction of propagation of the plane waves, it will be sufficient to take the case where the vibrations in these are rectilinear, as expressed by (18), and the axis of the gyrostat is parallel to the direction of vibration. If we put $P = R = 0$, the equations (108) reduce to

$$K\frac{d^2\xi}{dt^2} - Q\frac{d\zeta}{dt} = L, \quad K\frac{d^2\eta}{dt^2} = M, \quad K\frac{d^2\zeta}{dt^2} + Q\frac{d\xi}{dt} = N. \tag{125}$$

So far as the effect of the plane waves is concerned we may suppose that $\eta = 0$, $M = 0$. Also, assuming

$$rU_1 = Cz, \quad rU_1' = A'x + C'z, \tag{126}$$

we find $\xi = -A'f_1(ka), \quad \zeta = -\{C\psi_1(ka) + C'f_1(ka)\},$ (127)

$$L = -\tfrac{8}{3}\pi\mu ka^4 A'f_1'(ka), \quad N = -\tfrac{8}{3}\pi\mu ka^4\{C\psi_1'(ka) + C'f_1'(ka)\}. \tag{128}$$

Writing $Q = K\varpi$ (129)

and substituting in (125), we find

$$\{\tfrac{1}{5}\nu ka f_1(ka)+f_1'(ka)\}A'+\tfrac{1}{5}i\nu\frac{\varpi a}{c}f_1(ka)\,C'=-\tfrac{1}{5}i\nu\frac{\varpi a}{c}\psi_1(ka)\,C, \quad (130)$$

$$-\tfrac{1}{5}i\nu\frac{\varpi a}{c}f_1(ka)\,A'+\{\tfrac{1}{5}\nu ka f_1(ka)+f_1'(ka)\}\,C'$$
$$=-\{\tfrac{1}{5}\nu ka\,\psi_1(ka)+\psi_1'(ka)\}\,C, \quad (131)$$

whence

$$\left[\{\tfrac{1}{5}\nu ka\,f_1(ka)+f_1'(ka\}^2-\left\{\tfrac{1}{5}\nu\frac{\varpi a}{c}f_1(ka)\right\}^2\right]\frac{A'}{C}$$
$$=-i\nu\frac{\varpi a}{c}\{\psi_1(ka)\,f_1'(ka)-f_1(ka)\,\psi_1'(ka)\}$$
$$=i\nu\frac{\varpi a}{c}\,\frac{1}{k^4a^4}, \quad (132)$$

and

$$\left[\{\tfrac{1}{5}\nu ka\,f_1(ka)+f_1'(ka)\}^2-\left\{\tfrac{1}{5}\nu\frac{\varpi a}{c}f_1(ka)\right\}^2\right]\frac{C'}{C}$$
$$=-\Big[\tfrac{1}{5}\nu ka\,\psi_1(ka)+\psi_1'(ka)\}\{\tfrac{1}{5}\nu ka\,f_1(ka)+f_1'(ka)\}$$
$$-\nu^2\frac{\varpi^2a^2}{c^2}\psi_1(ka)\,f_1(ka)\Big]. \quad (133)$$

If we substitute from (126) in (41), the component displacements at a distance from the origin are given by

$$(u_1', v_1', w_1')=\frac{i}{k^2r}\left(\frac{C'y}{r},\ \frac{A'z-C'x}{r},\ -\frac{A'y}{r}\right)e^{ik(at-r)}. \quad (134)$$

For sufficiently small values of ϖ the value of C' will be sensibly the same as in the problem of § 6, whilst A' will be relatively small. The chief effect of the terms in A' will be to introduce a small vibration parallel to y at points on the axis of z, whilst at points on the axis of y the previously rectilinear vibration will become elliptic, the elliptic orbits being very elongated, with their major axes parallel to the axis of x.

Spherical Cavity.

9. By way of contrast with the problem of § 5, we may next consider the case where the inner boundary $r=a$ is free from stress; in other words, there is a spherical cavity in the medium. It

appears from (28) and (36) that the surface-conditions

$$\Sigma\,(F_n+F'_n)=0,\quad \Sigma\,(G_n+G'_n)=0,\quad \Sigma\,(H_n+H'_n)=0 \tag{135}$$

will be satisfied, provided

$$A_n\,(ka)\,S_n+C_n\,(ka)\,T_n+A'_n\,(ka)\,\frac{S'_n}{a^{2n+1}}+C'_n\,(ka)\,T'_n=0, \tag{136}$$

$$B_n\,(ka)\,S_n+D_n\,(ka)\,T_n+B'_n\,(ka)\,\frac{S'_n}{a^{2n+1}}+D'_n\,(ka)\,T'_n=0, \tag{137}$$

$$E_n\,(ka)\,U_n+E'_n\,(ka)\,U'_n=0. \tag{138}$$

Leaving aside the "free" vibrations, we have, when ka is small,

$$A_n\,(ka)=2\,(n-1),\quad B_n\,(ka)=\frac{k^2a^2}{2n+1}, \tag{139}$$

$$\left.\begin{aligned} C_n\,(ka)=\frac{2\,(n^2-1)}{1.3\ldots(2n-1)},\quad D_n\,(ka)=\frac{nk^2a^2}{1.3\ldots(2n+3)}\\ E'_n\,(ka)=\frac{n-1}{1.3\ldots(2n+1)}\end{aligned}\right\}, \tag{140}$$

$$A'_n\,(ka)=-\frac{k^2a^2}{2n+1},\quad B'_n\,(ka)=-2\,(n+2), \tag{141}$$

$$\left.\begin{aligned} C'_n(ka)&=-\frac{1.3\ldots(2n-3)(n+1)}{(ka)^{2n-1}}\\ D'_n(ka)&=\frac{1.3\ldots(2n+1)\,2n(n+2)}{(ka)^{2n+1}}\\ E'_n(ka)&=-\frac{1.3\ldots(2n-1)(n+2)}{(ka)^{2n+1}}\end{aligned}\right\}, \tag{142}$$

approximately. Substituting in (136), (137), (138), we find that when $S_n=0$, as in the case of incident plane waves,

$$T'_n=\frac{2\,(n^2-1)(ka)^{2n-1}}{\{1.3\ldots(2n-3)\}^2(2n-1)(2n^2+1)}\,T_n, \tag{143}$$

$$U'_n=\frac{(n-1)(ka)^{2n+1}}{\{1.3\ldots(2n-1)\}^2(2n+1)(n+2)}\,U_n. \tag{144}$$

The case $n = 1$ is exceptional, and demands a closer approximation. We find

$$A_1(ka) = -\tfrac{1}{3}k^2a^2, \quad B_1(ka) = \tfrac{1}{3}k^2a^2, \tag{145}$$

$$C_1(ka) = -\tfrac{2}{3}k^2a^2, \quad D_1(ka) = \tfrac{1}{15}k^2a^2, \quad E_1(ka) = -\tfrac{1}{15}k^2a^2, \tag{146}$$

$$A_1'(ka) = -\tfrac{1}{3}k^2a^2, \quad B_1'(ka) = -6, \tag{147}$$

$$C_1'(ka) = -\frac{2}{ka}, \quad D_1'(ka) = \frac{18}{k^3a^3}, \quad E_1'(ka) = -\frac{3}{k^3a^3}, \tag{148}$$

and thence $$T_1' = -\tfrac{2}{9}k^3a^3T_1, \quad U_1' = -\tfrac{1}{45}k^5a^5U_1. \tag{149}$$

Since, from (143), $$T_2' = \tfrac{2}{9}k^3a^3T_2, \tag{150}$$

it appears that the most important terms in the expressions for the scattered wave are those which arise from T_1' and T_2'. In the case of the wave-system represented by (18), we have

$$T_1' = -\tfrac{1}{9}k^3a^3\frac{y}{r}, \quad T_2' = \tfrac{1}{9}ik^4a^3\frac{xy}{r^2}; \tag{151}$$

the terms in question are therefore obtained by writing $B_1' = -\tfrac{2}{9}k^3a^3$, $B_2' = \tfrac{2}{9}k^3a^3$, in (43) and (47). And from (64) the dissipation-ratio is found to be

$$I = \tfrac{8}{27}\pi k^4a^6 + \tfrac{40}{81}\pi k^4a^6 = \tfrac{64}{81}\pi a^2(ka)^4, \tag{152}$$

approximately. This gives the usual law of intensity varying as the inverse fourth power of the wave-length.

Elastic Spherical Obstacle.

10. In the case of an elastic incompressible sphere of density ρ_0 and rigidity μ_0 embedded in an incompressible medium of density ρ and rigidity μ, we may write

$$c_0^2 = \mu_0/\rho_0, \tag{153}$$

and, since the time-factor (e^{ikct}) is everywhere the same,

$$k_0c_0 = kc, \tag{154}$$

whence $$\mu_0k_0^2 : \mu k^2 = \rho_0 : \rho. \tag{155}$$

The formulæ appropriate to the interior of the sphere will be of the same character as those of § 1, with k_0 written for k, and μ_0 for μ. The typical expressions for the displacements inside the sphere may therefore be taken to be

$$\left.\begin{aligned} u_n'' &= \frac{d}{dx}r^n S_n'' + (n+1)\psi_{n-1}(k_0 r)\frac{d}{dx}r^n T_n'' - nk_0^2 r^{2n+3}\psi_{n+1}(k_0 r)\frac{d}{dx}\frac{T_n''}{r^{n+1}} \\ &\qquad + \psi_n(k_0 r)\left(y\frac{d}{dz} - z\frac{d}{dy}\right) r^n U_n'' \\ v_n'' &= \frac{d}{dy}r^n S_n'' + (n+1)\psi_{n-1}(k_0 r)\frac{d}{dy}r^n T_n'' - nk_0^2 r^{2n+3}\psi_{n+1}(k_0 r)\frac{d}{dy}\frac{T_n''}{r^{n+1}} \\ &\qquad + \psi_n(k_0 r)\left(z\frac{d}{dx} - x\frac{d}{dz}\right) r^n U_n'' \\ w_n'' &= \frac{d}{dz}r^n S_n'' + (n+1)\psi_{n-1}(k_0 r)\frac{d}{dz}r^n T_n'' - nk_0^2 r^{2n+3}\psi_{n+1}(k_0 r)\frac{d}{dz}\frac{T_n''}{r^{n+1}} \\ &\qquad + \psi_n(k_0 r)\left(x\frac{d}{dy} - y\frac{d}{dx}\right) r^n U_n'' \end{aligned}\right\}, \tag{156}$$

The corresponding stresses F_n'', G_n'', H_n'' on a sphere of radius r will be as in (28), with μ_0 written for μ, k_0 for k, and S_n'', T_n'', U_n'' for S_n, T_n, U_n, respectively.

The continuity of displacement and stress at the surface $r = a$ requires

$$u_n + u_n' = u_n'', \quad v_n + v_n' = v_n'', \quad w_n + w_n' = w_n'', \tag{157}$$

$$F_n + F_n' = F_n'', \quad G_n + G_n' = G_n'', \quad H_n + H_n' = H_n''. \tag{158}$$

These lead in the usual manner to the formulæ

$$S_n + (n+1)\psi_{n-1}(ka)T_n + (n+1)f_{n-1}(ka)T_n' = S_n'' + (n+1)\psi_{n-1}(k_0 a)T_n'', \tag{159}$$

$$-nk^2a^2\psi_{n+1}(ka)T_n + \frac{S_n'}{a^{2n+1}} - nk^2a^2 f_{n+1}(ka)T_n' = -nk_0^2a^2\psi_{n+1}(k_0 a)T_n'', \tag{160}$$

$$\psi_n(ka)U_n + f_n(ka)U_n' = \psi_n(k_0 a)U_n'', \tag{161}$$

and

$$\mu A_n(ka)S_n + \mu C_n(ka)T_n + \mu A_n'(ka)\frac{S_n'}{a^{2n+1}} + \mu C_n'(ka)T_n'$$
$$= \mu_0 A_n(k_0 a)S_n'' + \mu_0 C_n(k_0 a)T_n'', \tag{162}$$

$$\mu B_n(ka)S_n + \mu D_n(ka)T_n + \mu B_n'(ka)\frac{S_n'}{a^{2n+1}} + \mu D_n'(ka)T_n'$$
$$= \mu_0 B_n(k_0 a)S_n'' + \mu_0 D_n(k_0 a)T_n'', \tag{163}$$

$$\mu E_n(ka)U_n + \mu E_n'(ka)U_n' = \mu_0 E_n(k_0 a)U_n'', \tag{164}$$

where the functions $A_n(ka)$, $A'_n(ka)$, &c., have the meanings assigned in (29), (30), (31), and (37), (38), (39).

When the extraneous disturbance consists of plane waves, we have $S_n = 0$. If, further, ka and k_0a be both small, the values of T'_n and U'_n in terms of T_n and U_n respectively may be approximated to without much difficulty. Thus, for $n = 1$, we find, making use of the formulæ (145) ... (148),

$$2T_1 + \frac{2}{ka} T'_1 = S''_1 + 2T''_1, \tag{165}$$

$$-\tfrac{1}{15}k^2a^2T_1 + \frac{S'_1}{a^3} - \frac{3}{k^3a^3} T'_1 = -\tfrac{1}{15}k_0^2a^2T''_1, \tag{166}$$

$$-\tfrac{2}{3}\mu k^2a^2T_1 - \tfrac{1}{3}\mu k^2a^2 \frac{S'_1}{a^3} - \frac{2\mu}{ka} T'_1 = -\tfrac{1}{3}\mu_0k_0^2a^2S''_1 - \tfrac{2}{3}\mu_0k_0^2a^2T''_1, \tag{167}$$

$$\tfrac{1}{15}\mu k^2a^2T_1 - 6\mu \frac{S'_1}{a^3} + \frac{18\mu}{k^3a^3} T_1 = \tfrac{1}{3}\mu_0k_0^2a^2S''_1 + \tfrac{1}{15}\mu_0k_0^2a^2T''_1, \tag{168}$$

where only the most important part of each coefficient has been retained.

The solution of these is facilitated by the consideration that T'_1 must be of the order k^3a^3 compared with T_1, as is seen from the results of § 9, which deals with a particular case of the present problem. Moreover T''_1 must be of the same order as T_1. We thence find, from (165) and (166),

$$S''_1 + 2T''_1 = 2T_1, \tag{169}$$

$$S'_1/a^3 = 3/k^3a^3 . T'_1. \tag{170}$$

Then (167) leads to the result

$$T'_1 = \tfrac{2}{9}ka \frac{\mu_0k_0^2a^2 - \mu k^2a^2}{\mu} T_1 = \tfrac{2}{9}k^3a^3 \left(\frac{\rho_0}{\rho} - 1\right) T_1, \tag{171}$$

by (155). This agrees with (149) if we put $\rho_0 = 0$.*

Similarly, the approximate forms of (161) and (164) are

$$\tfrac{1}{3}U_1 + \frac{1}{k^3a^3} U'_1 = \tfrac{1}{8}U''_1, \tag{172}$$

$$-\tfrac{1}{15}\mu k^2a^2U_1 - \frac{3\mu}{k^3a^3} U'_1 = -\tfrac{1}{15}\mu_0k_0^2a^2U''_1, \tag{173}$$

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* It is to be noticed that the equations (167) and (168) cannot be assumed to be sufficiently accurate for the determination of S''_1 and T''_1, owing to the approximate cancelling of the second and third terms in (168). The main interest lies, however, in the determination of T'_1.

leading to
$$U'_1 = \tfrac{1}{45}k^5a^5\left(\frac{\rho_0}{\rho}-1\right)U_1, \tag{174}$$

which also may be compared with (149).

It appears, on reference to (99), that the values of T'_1 and U'_1 are, to our present degree of approximation, the same as if the sphere were absolutely rigid. This might have been foreseen, for the terms in T_1'' represent, when k_0a is small, a mere translation of the sphere as a whole, whilst those in U_1'' represent a pure rotation.

For $n > 1$, I find

$$T'_n = -\frac{2(n^2-1)(\mu_0-\mu)}{2(n^2-1)\mu_0+(2n^2+1)\mu}\cdot\frac{(ka)^{2n-1}}{\{1.3\ldots(2n-3)\}^2(2n-1)}T_n, \tag{175}$$

$$U'_n = -\frac{(n-1)(\mu_0-\mu)}{(n-1)\mu_0+(n+2)\mu}\cdot\frac{(ka)^{2n+1}}{\{1.3\ldots(2n-1)\}^2(2n+1)}U_n. \tag{176}$$

These coincide with (143), (144), if we put $\mu_0 = 0$. In particular

$$T'_2 = -\frac{2(\mu_0-\mu)}{6\mu_0+9\mu}k^3a^3.T_2,\quad U'_2 = -\frac{\mu_0-\mu}{45(\mu_0+4\mu)}k^5a^5U_2.^* \tag{177}$$

The most important part of the disturbance at a distance in the scattered waves is therefore obtained by writing

$$B'_1 = \tfrac{2}{9}\left(\frac{\rho_0}{\rho}-1\right)k^3a^3,\quad B'_2 = -\frac{2(\mu_0-\mu)}{6\mu_0+9\mu}k^3a^3, \tag{178}$$

in the formulæ (43) and (47), respectively. The approximate value of the dissipation-ratio is

$$I = \left\{\tfrac{8}{27}\left(\frac{\rho_0}{\rho}-1\right)^2+\frac{40(\mu_0-\mu)^2}{(6\mu_0+9\mu)^2}\right\}\pi a^2(ka)^4. \tag{179}$$

Effect of Synchronism.

11. In the various problems above considered, general formulæ have been obtained for the disturbance produced in a train of plane waves by a spherical obstacle constituted in various ways; but no

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* When $(\rho_0-\rho)/\rho_0$ and $(\mu_0-\mu)/\mu$ are small, we have

$$T'_1 = \tfrac{2}{9}k^3a^3\frac{\Delta\rho}{\rho}T_1,\quad T'_2 = -\tfrac{2}{15}k^3a^3\frac{\Delta\mu}{\mu}T_2.$$

If we substitute the values of T_1 and T_2 from (25) and (26), and work out the expressions for the component *rotations* of the medium, in the scattered wave, we reproduce results given by Lord Rayleigh in a paper already cited (p. 120).

special examination has been made of cases where there is coincidence, or approximate coincidence, between the period of the waves and a natural period of vibration of the sphere. In this respect interest attaches chiefly to cases where the free period in question corresponds to a wave-length large in comparison with the circumference of the sphere, but our results will not be altogether dependent on this supposition.

The first instance where a free period of this character was met with was in the rectilinear vibrations of a rigid but movable sphere (§ 6) whose inertia bears a large ratio (κ) to that of an equal volume of the surrounding medium. If in (91) we put $S_1 = 0$, the accurate formula for the scattered waves of type T'_1 takes the form

$$\frac{T'_1}{T_1} = -\frac{(2\kappa+1)\,\psi_0(ka) - 9\psi_1(ka)}{(2\kappa+1)\,f_0(ka) - 9f_1(ka)} = -\frac{g(ka)}{G(ka) - ig(ka)}, \tag{180}$$

where

$$g(ka) = (2\kappa+1)\,\psi_0(ka) - 9\psi_1(ka), \quad G(ka) = (2\kappa+1)\,\Psi_0(ka) - 9\Psi_1(ka). \tag{181}$$

The modulus of the expression in the last member of (180) is never greater than unity; but it attains the value unity, and the amplitude of the scattered waves is therefore a maximum, when

$$G(ka) = 0. \tag{182}$$

If we substitute the values of $\Psi_0(ka)$, $\Psi_1(ka)$ from (4), this equation reduces to

$$\tan ka = \frac{2\kappa+1}{9}\,ka - \frac{1}{ka}. \tag{183}$$

The situation of the roots is easily found by a graphical construction, and it appears that when κ is large there is a small root given (to a first approximation) by

$$ka = \frac{3}{\sqrt{(2\kappa-8)}}; \tag{184}$$

a result which may be compared with (95). When (182) is satisfied exactly, we have

$$T'_1 = -iT_1. \tag{185}$$

Hence, putting $B'_1 = -i$ in (64), we have, for the dissipation-ratio,

$$I = \frac{6\pi}{k^2} = \frac{3}{2\pi}\lambda^2, \tag{186}$$

where λ is the wave-length.

A similar argument applies to the rotational vibrations. We find, from (92),

$$\frac{U_1'}{U_1} = -\frac{g(ka)}{G(ka) - ig(ka)}, \tag{187}$$

where

$$g(ka) = \tfrac{1}{5}\nu ka\psi_1(ka) + \psi_1'(ka), \quad G(ka) = \tfrac{1}{5}\nu ka\Psi_1(ka) + \Psi_1'(ka). \tag{188}$$

The wave-length of maximum dissipation is determined, as before, by $G(ka) = 0$, or

$$\tan ka = -\frac{1}{ka} - \frac{5ka}{\nu k^2a^2 - 15}. \tag{189}$$

The dissipation-ratio, when this is exactly satisfied, is given again by (186).

It is evident that the *form* of (180) or (187) is quite independent of the special hypotheses here made, or of the value of n. The general value of the dissipation-ratio I, when this is a maximum, is

$$I = \frac{2n+1}{2\pi}\lambda^2, \tag{190}$$

where n is the order of the spherical harmonic involved.

The wave-lengths of maximum scattering are in all cases sharply defined. It would be easy to institute calculations showing the effect of a slight deviation from a critical period; but this point has perhaps been sufficiently illustrated in the former paper.*

12. The particular example, just referred to, of a rigid sphere embedded in the medium is in one respect unfitted for the illustration of optical phenomena. In order to obtain a free period large compared with a/c, it was necessary to ascribe to the sphere a great relative inertia. It appears from § 5 that this implies a very considerable *general* scattering of incident waves, independently of any approximation to the critical wave-length. The dissipation, as given by (74), is, in fact, of the order $(\lambda/a)^4$ compared with what we know to be consistent with the actual degree of transparency of the atmosphere.†

The difficulty may be met, though perhaps rather artificially, if we have recourse to the more general suppositions of § 7. Let us suppose, for simplicity, that we have a rigid massless shell enclosing a nucleus attached to it by springs. Putting $\kappa = 0$ in (106), we have

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* *Proc. Lond. Math. Soc.*, Vol. xxxii., p. 18.

† Lord Rayleigh, *Phil. Mag.*, April, 1899.

$$\kappa' = \frac{\kappa_0 \sigma_0^2}{\sigma_0^2 - k^2 c^2}. \tag{191}$$

If we substitute this for κ in (94), we obtain the equation to determine the free periods, viz.,

$$\left(1 + \frac{2\kappa_0 \sigma_0^2}{\sigma_0^2 - k^2 c^2}\right) k^2 a^2 - 9ika - 9 = 0. \tag{192}$$

If the stiffness of the springs and the mass of the nucleus be so adjusted that $\sigma_0 a/c$ is small, this will be satisfied approximately by

$$k^2 a^2 = \frac{\sigma_0^2 a^2}{c^2} \div \left(1 + \tfrac{2}{9} \frac{\kappa_0 \sigma_0^2 a^2}{c^2}\right).^* \tag{193}$$

The wave-length of this free period is large compared with a, independently of the value of κ_0.

To determine the forced oscillations we find from (180), with κ' written for κ, and from (193),

$$\frac{T_1'}{T_1} = -\frac{2(\kappa'-1)}{(2\kappa'+1) k^2 a^2 - 9ika - 9} k^3 a^3, \tag{194}$$

provided the value of ka does not approximate to that given by (193). The dissipation-ratio will then be of the usual form

$$I = Ca^2 (ka)^4, \tag{195}$$

where C is a moderate numerical factor.

[Again, we might employ the hypothesis of an elastic sphere, considered in § 10. To obtain free periods with wave-lengths large compared with a, we must suppose the constants so adjusted that c_0, defined by (153), is small compared with c; in other words, we must have μ_0/ρ_0 small compared with μ/ρ. Since we are precluded, for the reason above given, from assigning any great relative inertia to the sphere, the only assumption left to us is to imagine that μ_0 is small compared with μ.

With molecules adjusted in this manner we might build up a model of a gaseous assemblage freely transparent to radiation generally, but with strong absorption for definite wave-lengths. It is to be noticed, however, that in reconciling these requirements we have made it impossible to get rid of that constituent of the general scattered light which is dependent on the harmonic T_2' and is represented by the formulæ (47).* If we survey the various problems of this paper, we find that this constituent is always important (with an amplitude proportional to k^2a^3/r) except in the case of § 10, where it can be annulled (approximately) by making $\mu_0 = \mu$. This is, however, inadmissible for the reason just explained.]†

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* This indicates that the frequency of vibration of the nucleus is less than if the containing shell were held fixed, in the ratio $\left(1 + \tfrac{2}{9} \frac{\kappa_0 \sigma_0^2 a^2}{c^2}\right)^{-\frac{1}{2}}$.

Thursday, April 5th, 1900.

Lt.-Col. A. J. C. CUNNINGHAM, R.E., Vice-President, in the Chair.

Nineteen members present.

Mr. F. W. B. Frankland, B.A., Fellow of Clare College, Cambridge, was elected a member.

The following persons were admitted into the Society:—Miss B. M. Cave Browne Cave, Miss F. E. Cave Browne Cave, and Mr. R. W. H. T. Hudson.

Mr. Tucker read a letter from Mrs. J. J. Walker, thanking the Society for its vote of condolence with her on the loss of her husband.

Mr. Macdonald read a note "On the Addition-Theorem for the Bessel Functions," and Mr. Basset made a communication on "The Orthoptic Loci of Curves of a given class."

Prof. Love gave a proof of "The Uniform Convergence of Fourier's Series."

Major MacMahon spoke on "An Extension of Orthogonal and Boolian Covariants."

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* For the argument that this constituent is absent from the light scattered by finely divided particles, see Lord Rayleigh, *Scientific Papers*, Vol. I., p. 107.

† *Added Sept., 1900.*

Messrs. Macaulay, R. W. H. T. Hudson, Whittaker, and the Chairman spoke to one or more of the papers.

A paper by Mr. Bromwich, entitled "Note on Weierstrass's Reduction of a Family of Bilinear Forms," was received.

The Chairman announced that the May meeting would be made "special," as announced by letter to the members of the Society.

The following presents were made to the Library:—

"Educational Times," April, 1900.

"Indian Engineering," Vol. XXVII., Nos. 7–10, Feb. 17–Mar. 10, 1900.

"Nautical Almanac for 1903," 8vo; Edinburgh, 1900. From Nautical Almanac Office.

"Mathematical Gazette," Vol. I., No. 20; 1900.

"Periodico di Matematica," Serie 2, Vol. II., Fasc. 5; Livorno, 1900.

"Supplemento al Periodico di Matematica," Anno III., Fasc. 5; Livorno, 1900.

Darboux, G.—"Sur la Déformation des Surfaces du second degré," 4to; Paris, 1900.

Ostwald, W.—"Periodische Erscheinungen bei der Auflösung des Chroms in Säuren," 8vo; Leipzig, 1900.

Rice, H.—"The Theory and Practice of Interpolation," 8vo; Lynn, Mass., 1900. From the author.

"Transactions of the American Mathematical Society," Vol. I., No. 1; 1900.

"Memoirs of the National Academy of Sciences," Vol. VIII., Memoir 4; Washington, 1899.

"Grundlage der Geometrie" (3 pp.), and "Euklidische Grundlage der Geometrie" (7 pp.), from the author, Dr. W. Veltmann.

"Report on the Progress of the Solution of the Problem of Three Bodies," by E. T. Whittaker (reprint from "British Association Report," 1899).

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXVI., Nos. 426, 427; 1900.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXIV., St. 2, 3; Leipzig, 1900.

"Annales de la Faculté des Sciences de Toulouse," Série 2, Tome I., 3^{me} Fascicule; 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. VI., Nos. 5, 6, Feb.-Mar., 1900; New York.

"Bulletin des Sciences Mathématiques," Tome XXIV., Jan., 1900; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VI., Nos. 1, 2; Napoli, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. IX., Fasc. 4, 5; Roma, 1900.

"Nyt Tidsskrift for Matematik," B. Aargang XI., Nr. 1; Copenhagen, 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 46, 47; Nov., 1897.

The Addition-Theorem for the Bessel Functions. *By* H. M. MACDONALD. Read and received April 5th, 1900.

The addition-theorems for $J_0(R)$ and $Y_0(R)$, where

$$R^2 = r^2 + r_1^2 - 2rr_1 \cos\theta,$$

have been given by Neumann, the addition-theorem for $\dfrac{J_n(R)}{R^n}$ by Gegenbauer and Sonine, and that for $K_{\frac{1}{2}}(\iota R)/R^{\frac{1}{2}}$ by Heine. The object of the following paper is the development of a method by which the addition-theorems for $\dfrac{J_n(R)}{R^n}$ and $\dfrac{K_n(\iota R)}{R^n}$ can be obtained. In the first section an integral, which is suggested by the integral obtained in a previous paper (*Proceedings*, Vol. xxx., p. 170), for the product $K_n(a)\,K_n(b)$ is discussed. In the second section this integral is employed to obtain the addition-theorems.

1. *The Integral* $\int e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\, I_n\left(\frac{ab}{t}\right)\frac{dt}{t}$.

Writing
$$u = \int_{t_0}^{t_1} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\, I_n\left(\frac{ab}{t}\right)\frac{dt}{t},$$

where the path of integration is subjected to the condition that, if it is not closed, the points at which it ends are independent of a and b, it follows that

$$\frac{\partial^2 u}{\partial a^2} + \frac{1}{a}\frac{\partial u}{\partial a} + \left(1 - \frac{n^2}{a^2}\right)u$$
$$= \int_{t_0}^{t_1} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\frac{dt}{t}\left\{\left(\frac{a^2}{t^2} - \frac{2}{t} + 1 - \frac{n^2}{a^2}\right) I_n\left(\frac{ab}{t}\right)\right.$$
$$\left. + \left(\frac{b}{at} - \frac{2ab}{t^2}\right) I_n'\left(\frac{ab}{t}\right) + \frac{b^2}{t^2} I_n''\left(\frac{ab}{t}\right)\right\},$$

provided, as will subsequently appear, that this integral is convergent. Now

$$I_n''\left(\frac{ab}{t}\right) + \frac{t}{ab} I_n'\left(\frac{ab}{t}\right) - \left(1 + \frac{t^2n^2}{a^2b^2}\right) I_n\left(\frac{ab}{t}\right) = 0;$$

therefore

$$\frac{\partial^2 u}{\partial a^2}+\frac{1}{a}\frac{\partial u}{\partial a}+\left(1-\frac{n^2}{a^2}\right)u$$

$$=\int_{t_0}^{t_1} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\frac{dt}{t}\left\{\left(\frac{a^2+b^2}{t^2}-\frac{2}{t}+1\right)I_n\left(\frac{ab}{t}\right)-\frac{2ab}{t^2}I_n'\left(\frac{ab}{t}\right)\right\};$$

that is,

$$\frac{\partial^2 u}{\partial u^2}+\frac{1}{a}\frac{\partial u}{\partial a}+\left(1-\frac{n^2}{a^2}\right)u=2e^{\frac{1}{2}t-[(a^2+b^2)/2t]}I_n\left(\frac{ab}{t}\right)\frac{1}{t}\Big|_{t_0}^{t_1}.$$

Now the expression on the right-hand side vanishes, provided that the real parts of $(a\pm b)^2$ are positive, the real part of n is greater than -1, and t_0, t_1 have for values two of the quantities 0, $c+\infty\iota$, $c-\infty\iota$, where c is some real positive quantity; further, all the integrals involved in the above process are then convergent; hence, denoting by u_1, u_2, u_3 the three functions u which correspond to the three sets of values of t_0 and t_1,

$$t_0=c-\infty\iota,\qquad t_1=c+\infty\iota,$$

$$t_0=c-\infty\iota,\qquad t_1=0,$$

$$t_0=0,\qquad t_1=c+\infty\iota,$$

any one of them satisfies the differential equation

$$\frac{d^2u}{da^2}+\frac{1}{a}\frac{du}{da}+\left(1-\frac{n^2}{a^2}\right)u=0,$$

and also, from the symmetry of the expression, the equation

$$\frac{d^2u}{db^2}+\frac{1}{b}\frac{du}{db}+\left(1-\frac{n^2}{b^2}\right)u=0.$$

Therefore $\quad u=f_1(b)\,J_n(a)+f_2(b)\,J_{-n}(a),$

where, since u satisfies the second differential equation above,

$$f_1(b)=AJ_n(b)+BJ_{-n}(b),$$

$$f_2(b)=A'J_n(b)+B'J_{-n}(b),$$

and hence

$$u=J_n(a)\left\{AJ_n(b)+BJ_{-n}(b)\right\}+J_{-n}(a)\left\{A'J_n(b)+B'J_{-n}(b)\right\}.$$

It remains to determine the values of the constants A, B, A', B' belonging to each of the functions u_1, u_2, u_3. Taking first the integral

$$u_1=\int_{c-\infty\iota}^{c+\infty\iota}e^{\frac{1}{2}t-[(a^2+b^2)/2t]}I_n\left(\frac{ab}{t}\right)\frac{dt}{t},$$

it appears that $\frac{u_1}{b^n}$ tends to a finite limit as b tends to zero; hence

$$B = B' = 0,$$

and therefore $$u_1 = J_n(b)\,\{AJ_n(a) + A'J_{-n}(a)\}.$$

Now $$\mathrm{Lt}_{b=0}\frac{u_1}{b^n} = \frac{a^n}{2^n\,\Pi(n)}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-(a^2/2t)}\,\frac{dt}{t^{n+1}},$$

that is, $$\mathrm{Lt}_{b=0}\frac{u_1}{b^n} = \frac{2\pi\iota}{2^n\,\Pi(n)}\,J_n(a),$$

but also $$\mathrm{Lt}_{b=0}\frac{u_1}{b^n} = \frac{1}{2^n\,\Pi(n)}\,\{AJ_n(a) + A'J_{-n}(a)\};$$

hence $$A' = 0, \quad A = 2\pi\iota,$$

and therefore $$u_1 = 2\pi\iota\,J_n(a)\,J_n(b);$$

that is, $$J_n(a)\,J_n(b) = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-\{(a^2+b^2)/2t\}}\,I_n\left(\frac{ab}{i}\right)\frac{dt}{t}.$$

This integral can immediately be transformed into the integral given by Sonine as follows :—

$$J_n(a)\,J_n(b)$$
$$= \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\,\frac{dt}{t^{n+1}}\int_0^\pi \frac{a^n b^n}{2^n\,\Pi\left(n-\frac{1}{2}\right)\sqrt{\pi}}\,e^{(ab\cos\theta)/t}\sin^{2n}\theta\,d\theta;$$

whence, changing the order of integration and integrating,

$$J_n(a)\,J_n(b) = \frac{a^n b^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}\int_0^\pi \frac{J_n\{\sqrt{(a^2+b^2-2ab\cos\theta)}\}}{\{\sqrt{(a^2+b^2-2ab\cos\theta)}\}^n}\sin^{2n}\theta\,d\theta.$$

In the case of u_2, suppose that $|\,a\,| < |\,b\,|$; then $\frac{u_2}{a^n}$ tends to a finite limit as a tends to zero, for in the neighbourhood of $t = 0$ the integrand tends to zero, and at every other point of the path of integration to a finite limit; therefore

$$A' = B' = 0;$$

and hence $$u_2 = J_n(a)\,\{AJ_n(b) + BJ_{-n}(b)\}.$$

Now $$\mathrm{Lt}_{a=0}\frac{u_2}{a^n} = \mathrm{Lt}_{a=0}\frac{1}{a^n}\int_{c-\infty\iota}^{0} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\,I_n\left(\frac{ab}{t}\right)\frac{dt}{t};$$

that is, $$\mathrm{Lt}_{a=0}\frac{u_2}{a^n} = \frac{b^n}{2^n\,\Pi(n)}\int_{c-\infty\iota}^{0} e^{\frac{1}{2}t-b^2/2t}\,\frac{dt}{t^{n+1}},$$

or $$*\mathrm{Lt}_{a=0}\frac{u_2}{a^n} = -\frac{\pi}{2^n\,\Pi(n)\sin n\pi}\left\{J_{-n}(b) - e^{n\pi\iota}J_n(b)\right\};$$

but $$\mathrm{Lt}_{a=0}\frac{u_2}{a^n} = \frac{1}{2^n\,\Pi(n)}\left\{AJ_n(b) + BJ_{-n}(b)\right\};$$

therefore $$u_2 = -\frac{\pi}{\sin n\pi}\left\{J_{-n}(b) - e^{n\pi\iota}J_n(b)\right\}J_n(a).$$

Hence it follows that

$$-\frac{\pi}{\sin n\pi}\left\{J_{-n}(b) - e^{n\pi\iota}J_n(b)\right\}J_n(a) = \int_{c-\infty\iota}^{0} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\,I_n\left(\frac{ab}{t}\right)\frac{dt}{t},$$

that is, $$J_n(a)\,K_n(\iota b) = -\frac{e^{-n\pi\iota}}{2}\int_{c-\infty\iota}^{0} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\,I_n\left(\frac{ab}{t}\right)\frac{dt}{t},$$

where $|a| < |b|$. Similarly it can be shown that

$$\frac{\pi}{\sin n\pi}\left\{J_{-n}(b) - e^{-n\pi\iota}J_n(b)\right\}J_n(a) = \int_{0}^{c+\infty\iota} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\,I_n\left(\frac{ab}{t}\right)\frac{dt}{t},$$

where $|a| < |b|$. The relation given in the *Proceedings*, Vol. xxx., p. 170, and other similar relations, can be obtained in an analogous manner.

2. *The Addition-Theorems.*

The addition-theorem for $\dfrac{J_n(R)}{R^n}$ can be at once deduced from the integral relation first established as follows:—

The function $\dfrac{J_n(R)}{R^n}$ is given by the relation

$$\frac{J_n(R)}{R^n} = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-R^2/2t}\,\frac{dt}{t^{n+1}};$$

that is, writing $$R^2 = a^2+b^2-2ab\cos\theta,$$

$$\frac{J_n\left\{\sqrt{(a^2+b^2-2ab\cos\theta)}\right\}}{\left\{\sqrt{(a^2+b^2-2ab\cos\theta)}\right\}^n} = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-[(a^2+b^2)/2t]+(ab/t)\cos\theta}\,\frac{dt}{t^{n+1}};$$

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* *Proceedings*, Vol. xxix., p. 110.

Now* $e^{(ab/t)\cos\theta} = 2^{n-1}\,\Pi\,(n-1) \sum_{k=0}^{k=\infty} (n+k)\, C_k^n(\cos\theta)\, \dfrac{J_{n+k}\left(\dfrac{ab\iota}{t}\right)}{\left(\dfrac{ab}{t}\right)^n \iota^{n+k}}$,

or $\quad e^{(ab/t)\cos\theta} = 2^{n-1}\,\Pi\,(n-1) \sum_{k=0}^{k=\infty} (n+k)\, C_k^n(\cos\theta)\, \dfrac{I_{n+k}\left(\dfrac{ab}{t}\right)}{\left(\dfrac{ab}{t}\right)^n}$,

where n is not zero and $C_k^n(\cos\theta)$ is the coefficient of h^k in the expansion of $(1-2h\cos\theta+h^2)^{-n}$ in powers of h. Hence it follows that

$$\frac{J_n\{\surd(a^2+b^2-2ab\cos\theta)\}}{\{\surd(a^2+b^2-2ab\cos\theta)\}^n}$$
$$= 2^{n-1}\,\Pi\,(n-1) \sum_{k=0}^{k=\infty} (n+k)\, \frac{C_k^n(\cos\theta)}{a^n b^n}\, \frac{1}{2\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\, I_{n+k}\left(\frac{ab}{t}\right) \frac{dt}{t},$$

$n \neq 0$, that is, since

$$\frac{1}{2\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-[(a^2+b^2)/2t]}\, I_{n+k}\left(\frac{ab}{t}\right) \frac{dt}{t} = J_{n+k}(a)\, J_{n+k}(b),$$

$$\frac{J_n\{\surd(a^2+b^2-2ab\cos\theta)\}}{\{\surd(a^2+b^2-2ab\cos\theta)\}^n}$$
$$= 2^n\,\Pi\,(n-1) \sum_{k=0}^{k=\infty} (n+k)\, \frac{J_{n+k}(a)}{a^n}\, \frac{J_{n+k}(b)}{b^n}\, C_k^n(\cos\theta),$$

$n \neq 0$. When $n = 0$, it can be easily shown that

$$e^{(ab/t)\cos\theta} = I_0\left(\frac{ab}{t}\right) + 2 \sum_{k=1}^{k=\infty} I_k\left(\frac{ab}{t}\right) \cos k\theta,$$

and hence, as above, that

$$J_0\{\surd(a^2+b^2-2ab\cos\theta)\} = J_0(a)\, J_0(b) + 2 \sum_1^\infty J_k(a)\, J_k(b) \cos k\theta.$$

These results agree with those given by Neumann, Gegenbauer, and Sonine.

The addition-theorem for $\dfrac{K_n(\iota R)}{R^n}$ is obtained as follows:—From the relation

$$\frac{J_{-n}(R) - e^{n\pi\iota} J_n(R)}{2\iota \sin n\pi\, R^n} = -\frac{1}{2\pi\iota} \int_{c-\infty\iota}^{0} e^{\frac{1}{2}t-[(a^2+b^2-2ab\cos\theta)]/2t}\, \frac{dt}{t^{n+1}},$$

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* Sonine, *Math. Ann.*, Vol. XVI.

where the real part of n is greater than -1, it follows that

$$-\frac{\pi}{\sin n\pi}\left\{\frac{J_{-n}(R)}{R^n}-e^{n\pi\iota}\frac{J_n(R)}{R^n}\right\}$$

$$=2^n\Pi(n-1)\sum_{k=0}^{k=\infty}(n+k)\frac{C_k^n(\cos\theta)}{a^nb^n}\int_{c=\infty\iota}^{0}e^{\frac{1}{2}t-[(a^2+b^2)/2t]}I_{n+k}\left(\frac{ab}{t}\right)\frac{dt}{t},$$

$n \neq 0$; that is, by § 1,

$$\frac{\pi}{\sin n\pi}\left\{\frac{J_{-n}(R)}{R^n}-e^{n\pi\iota}\frac{J_n(R)}{R^n}\right\}$$

$$=2^n\Pi(n-1)\sum_{k=0}^{k=\infty}(n+k)\frac{\pi C_k^n(\cos\theta)}{\sin(n+k)\pi}$$

$$\times\left\{\frac{J_{-n-k}(b)}{b^n}-e^{(n+k)\pi\iota}\frac{J_{n+k}(b)}{b^n}\right\}\frac{J_{n+k}(a)}{a^n},$$

where $|a| < |b|$, or, as it may be written,

$$\frac{K_n\{\iota\sqrt{(a^2+b^2-2ab\cos\theta)}\}}{\{\sqrt{(a^2+b^2-2ab\cos\theta)}\}^n}$$

$$=2^n\Pi(n-1)\sum_{k=0}^{k=\infty}e^{\frac{1}{2}(k\pi\iota)}(n+k)\frac{K_{n+k}(\iota b)}{b^n}\frac{J_{n+k}(a)}{a^n}C_k^n(\cos\theta).$$

This includes, as a particular case ($n=\frac{1}{2}$), Heine's result

$$\frac{1}{\sqrt{(2\pi)}}\frac{K_{\frac{1}{2}}\{\iota\sqrt{(a^2+b^2-2ab\cos\theta)}\}}{\{\sqrt{(a^2+b^2-2ab\cos\theta)}\}^{\frac{1}{2}}}$$

$$\equiv\frac{e^{-\iota\sqrt{(a^2+b^2-2ab\cos\theta)}}}{\sqrt{(a^2+b^2-2ab\cos\theta)}}=\sum_{k=0}^{k=\infty}e^{\frac{1}{2}(k\pi\iota)}(k+\tfrac{1}{2})\frac{K_{k+\frac{1}{2}}(\iota b)}{b^{\frac{1}{2}}}\frac{J_{k+\frac{1}{2}}(a)}{a^{\frac{1}{2}}}P_n(\cos\theta),$$

and, when $n = 0$, it takes the form

$$K_0\{\iota\sqrt{(a^2+b^2-2ab\cos\theta)}\}$$

$$=K_0(\iota b)\,J_0(a)+2\sum_{1}^{\infty}e^{\frac{1}{2}(k\pi\iota)}K_k(\iota b)\,J_k(a)\cos k\theta,$$

which is equivalent to Neumann's result. Addition-theorems for any other solution of Bessel's equation can be obtained in a similar way.

Note on Weierstrass's Reduction of a Family of Bilinear Forms.

By T. J. I'A. Bromwich. Received April 2nd, 1900, and communicated April 5th, 1900.

In Weierstrass's original memoir (*Berliner Berichte*, 1868, p. 310; *Gesammelte Werke*, Vol. II., p. 19) the method of reduction of the family (*Schaar*) of forms $(rA-B)$ requires that the determinants $|A|$, $|B|$ should not be zero. In case $|A| = 0$, or $|B| = 0$ while $|rA-B| \not\equiv 0$, Weierstrass replaces the fundamental forms A, B by others, $gA+hB$, $g'A+h'B$, whose determinants are not zero. Stickelberger [*Crelle*, Vol. LXXXVI. (1878), p. 20] has shown that in dealing with the non-infinite roots of $|rA-B| = 0$ the restriction $|A| \neq 0$ is unnecessary; he has not, however, considered what modification of the general process is required in order to include the effect of the infinite roots.

In what follows a method of investigation is given which provides a new proof of part of the Weierstrass-Stickelberger investigation, as well as indicating the terms which have to be added on account of the infinite roots of $|rA-B| = 0$; these terms are (as might be anticipated) the same as those given by the zero roots of $|A-sB| = 0$ (*cf.* a previous paper, *Proc. Lond. Math. Soc.*, p. 78 above, § 3).

If $|A| = 0$, the coefficient of $x_\alpha y_\beta$ in the reciprocal bilinear form $(rA-B)^{-1}$ will be a fraction in r, the numerator being in general of higher degree (or at least not of lower degree) than the denominator. Thus, on expanding in powers of $(1/r)$, we can write

$$(rA-B)^{-1} = L_m r^m + \dots + L_1 r + L_0 + M_1 (1/r) + M_2 (1/r)^2 + \dots,$$

where $m \geqq 0$, and the coefficients of the powers of r will be various bilinear forms. Thus, multiplying by $(rA-B)$, we find that

$$E = (rA-B)(L_m r^m + \dots) = (L_m r^m + \dots)(rA-B).$$

Now equate coefficients of r, $(1/r)$, and the term independent of r; then

$$AL_0 - BL_1 = 0 = L_0 A - L_1 B, \quad AM_2 - BM_1 = 0 = M_2 A - M_1 B,$$

$$AM_1 - BL_0 = E = M_1 A - L_0 B.$$

Hence $$AM_1 A = A(E + L_0 B) = A + BL_1 B,$$

$$AM_2 A = AM_1 B = B + BL_0 B,$$

or $$A = AM_1A - BL_1B, \quad B = AM_2A - BL_0B.$$

In these forms for A, B the terms AM_1A, AM_2A are respectively the coefficients of $(1/r), (1/r)^2$ in the expansion of $A\,(rA-B)^{-1}A$ in powers of $(1/r)$. Now $A\,(rA-B)^{-1}A$ can be split up into partial fractions corresponding to each finite root of $|\,rA-B\,| = 0$, just as in Weierstrass's investigation;* and each fraction can be expanded in powers of $(1/r)$. So that Weierstrass's reduced forms for A, B hold even if $|\,A\,| = 0$, $|\,B\,| = 0$, so far as the finite roots are concerned (*cf.* Stickelberger, *l.c.*, pp. 34–37). It follows that the infinite roots of $|\,rA-B\,| = 0$ must naturally correspond to the terms $-BL_1B$, $-BL_0B$ in A and B. Now BL_1B, BL_0B are respectively the coefficient of r and the term independent of r in the expansion of $B\,(rA-B)^{-1}B$ in powers of $(1/r)$. It will be proved that these terms can also be reduced to forms similar to the Weierstrassian types; we now proceed to the evaluation of these forms.

We can expand the form $B\,(rA-B)^{-1}B$ in the way given by Weierstrass (*l.c.*), Darboux (*Liouville*, Vol. xix., Sér. 2, 1874, p. 347), or Stickelberger (*l.c.*) as a sum of terms of the type

$$-U_aV_a/w_{a-1}w_a,$$

using Stickelberger's notation. In order to make the results perfectly clear, I write out the complete meanings of our symbols:—

We have $$B\,(rA-B)^{-1}B = -W_0/w_0,$$

where $$W_{a-1} = \begin{vmatrix} ra_{11}-b_{11}, & \dots, & ra_{1n}-b_{1n}, & v_1', & \dots, & v_1^{a-1}, & \dfrac{\partial B}{\partial y_1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ ra_{n1}-b_{n1}, & \dots, & ra_{nn}-b_{nn}, & v_n', & \dots, & v_n^{a-1}, & \dfrac{\partial B}{\partial y_n} \\ u_1', & \dots, & u_n', & 0, & \dots, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ u_1^{a-1}, & \dots, & u_n^{a-1}, & 0, & \dots, & 0, & 0 \\ \dfrac{\partial B}{\partial x_1}, & \dots, & \dfrac{\partial B}{\partial x_n}, & 0, & \dots, & 0, & 0 \end{vmatrix},$$

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* Weierstrass does not introduce the condition $|\,A\,| \neq 0$ in breaking up $A\,(rA-B)^{-1}A$ into partial fractions, so that his process is legitimate here. He makes the restriction $|\,A\,| \neq 0$ in showing that A, B will be the coefficients of $(1/r)$, $(1/r)^2$ in the expansion of these partial fractions.

U_{α} is the value of $W_{\alpha-1}$ with the last column replaced by

$$v_1^{\alpha}, \ldots, v_n^{\alpha}, 0, \ldots, 0;$$

V_{α} the value of $W_{\alpha-1}$ with the last row replaced by

$$u_{\alpha}, \ldots, u_n^{\alpha}, 0, \ldots, 0;$$

w_{α} is the value of $W_{\alpha-1}$ when both the last row and the last column are so replaced. For uniformity $w_0 = | rA - B |$, and W_0 is $W_{\alpha-1}$ without any u's or v's.

The u's and v's are arbitrary constants to be determined in the step-by-step way indicated by Stickelberger; of course this has to be appropriately modified here, as the root of $w_0 = 0$ in question is infinite. The rule will accordingly be as follows:—Let the index of the *highest* power of r in W_0 be λ_1; then choose $u_1', \ldots, u_n', v_1', \ldots, v_n'$, so that the coefficient of the corresponding power of r in w_1 does not vanish; having determined these, consider W_1; let the index of the highest power of r in W_1 be λ_2; then choose $u_1^2, \ldots, u_n^2, v_1^2, \ldots, v_n^2$, so that the coefficient of this power in w_2 does not vanish; keeping $u_{\beta}', u_{\beta}^2, v_{\beta}', v_{\beta}^2$ the same, let the index of the highest power of r in W_2 be λ_3; then u_{β}^3, v_{β}^3 are to be chosen so that the coefficient of this power in w_3 is not zero; and so on until all the constants are determined. For uniformity, λ_0 is the index of the highest power of r in w_0.

We follow Stickelberger's process closely, and it will be sufficient to state the main results, pointing out the differences from his investigations. Just as proved by him, we shall have that W_{α} is not identically zero, if $\alpha < (n+1)$. Taking the identity [Stickelberger's (1), § 1, *l.c.*]

$$W_{\alpha} w_{\alpha-1} = W_{\alpha-1} w_{\alpha} - U_{\alpha} V_{\alpha},$$

we note that here the highest power of r on the left is the $(\lambda_{\alpha-1} + \lambda_{\alpha+1})^{\text{th}}$, and that on the right is *at most* the $(2\lambda_{\alpha})^{\text{th}}$. Hence

$$\lambda_{\alpha-1} + \lambda_{\alpha+1} \overset{=}{<} 2\lambda_{\alpha},$$

i.e.,
$$\lambda_{\alpha} - \lambda_{\alpha-1} \overset{=}{>} \lambda_{\alpha+1} - \lambda_{\alpha}.$$

It will be convenient to write $e_{\alpha} = 1 + \lambda_{\alpha} - \lambda_{\alpha-1}$ (the reason for adding unity will appear below), and then the inequality is

$$e_{\alpha} \overset{=}{>} e_{\alpha+1}.$$

Under ordinary circumstances $\lambda_{\alpha} = n - \alpha$, and then $e_{\alpha} = 0$. It is *clear that* $\lambda_{\alpha} \not> (n-\alpha)$, and, if $(n-\alpha) > \lambda_{\alpha}$, we have $(n-\alpha) - \lambda_{\alpha}$ infinite roots of every a^{th} minor; calling this number μ_a, we now have $e_a = \mu_{a-1} - \mu_a$. Suppose that we take the series $\lambda_n, \lambda_{n-1}, \ldots$, and let the first λ which does *not* satisfy the condition $\lambda_a = n - a$ be $\lambda_{\kappa-1}$; then $\lambda_{\kappa-1} < n-(\kappa-1)$ (as is obvious from definition of $\lambda_{\kappa-1}$), and

$$e_\kappa = 1 + \lambda_\kappa - \lambda_{\kappa-1} = n - \kappa + 1 - \lambda_{\kappa-1} > 0$$

Hence $\quad e_1 \overset{=}{>} e_2 \overset{=}{>} e_3 \ldots \overset{=}{>} e_\kappa > 0, \quad e_{\kappa+1} = e_{\kappa+2} = \ldots = e_n = 0,$

and, further, we have

$$\mu_0 > \mu_1 > \mu_2 \ldots > \mu_{\kappa-1} > 0, \quad \mu_\kappa = \mu_{\kappa+1} = \ldots = 0.$$

It follows that the μ's follow the lines of Stickelberger's λ's, and the e's behave in the same way as the indices of ordinary (finite) invariant-factors (*Elementartheiler*).

Now, returning to the identity

$$-\frac{W_0}{w_0} = -\Sigma \frac{U_a V_a}{w_{a-1} w_a},$$

we can write $-w_{a-1} w_a / r^{\lambda_{a-1}+\lambda_a}$ in the form of a series of powers of $(1/r)$, starting with a constant term; and this can be divided into the product of two similar series p_a, q_a.

Now write $\quad U_a/p_a = r^{\lambda_a} [X_1 + X_2/r + X_3/r^2 + \ldots],$

$$V_a/q_a = r^{\lambda_a} [Y_1 + Y_2/r + Y_3/r^2 + \ldots],$$

where $X_1, X_2, \ldots$ are linear functions of $x_1, \ldots, x_n$; and $Y_1, Y_2, \ldots$ of $y_1, \ldots, y_n$. Then we have

$$\begin{aligned} -U_a V_a/(w_{a-1} w_a) &= r^{(\lambda_a - \lambda_{a-1})} [X_1 + X_2/r + \ldots][Y_1 + Y_2/r + \ldots] \\ &= r^{(e_a - 1)} [X_1 + X_2/r + \ldots][Y_1 + Y_2/r + \ldots]. \end{aligned}$$

The quantities we require are the coefficient of r, and the term independent of r in the expansion of

$$-W_0/w_0 = -\Sigma U_a V_a/(w_{a-1} w_a).$$

Thus the parts of BL_0B, BL_1B from the term $-U_a V_a / w_{a-1} w_a$ will be

$$\left.\begin{array}{ll} (\text{in } BL_0B), & X_1 Y_e \;\; + X_2 Y_{e-1} + \ldots + X_e Y_1 \\ (\text{in } BL_1B), & X_1 Y_{e-1} + X_2 Y_{e-2} + \ldots + X_{e-1} Y_1 \end{array}\right\} \quad (e = e_a).$$

Hence, finally, the parts of A, B (in their reduced forms) which belong to the root $r = \infty$ of $| rA - B |$ will be

$$A, \quad -\Sigma\,(X_1 Y_{e-1} + \ldots + X_{e-1} Y_1),$$
$$B, \quad -\Sigma\,(X_1 Y_e + \ldots + X_e Y_1),$$

the summations extending to all the e's which differ from zero.

Again, following Stickelberger, it may be proved without difficulty that λ_a is the index of the highest power of r that occurs in *any* a^{th} minor of $| rA - B |$; and thus our problem is completely solved (*cf.* Stickelberger, § 2, Satz, I., II., III.).

In conclusion it may be pointed out that the reduced parts so arrived at are precisely the same as would be found from the zero roots of $| sB - A | = 0$, a fact which may be seen by inspection of our forms for the X's and Y's.

[*Note added* 11*th May*, 1900.—Since the above was completed I have received from Herr Alfred Loewy an advance copy of his valuable paper in *Crelle*, Vol. CXXII., p. 53, "Ueber Scharen reeller quadratischer und Hermitescher Formen." Applying our results, we obtain the following theorem, deduced from the one in § 1 of Loewy's paper by including the effect of infinite invariant-factors.

If A, B are real quadratic forms (whose determinants may or may not vanish), and if q' denote the *characteristic** of B, then the indices of the invariant-factors of $| rA - B |$ must satisfy the inequality

$$q' \geqq s + \Sigma\, E\,(\tfrac{1}{2}h) + \Sigma\, E\,\{\tfrac{1}{2}\,(l-1)\},$$

where $2s$ is the sum of the indices of the invariant-factors corresponding to complex roots of $| rA - B | = 0$; h represents the index of any invariant-factor corresponding to a real root ($r = 0$ excluded, but $r = \infty$ included, the indices for $r = \infty$ being the e's as found above); l is the index of any invariant-factor to base r. The symbols $E\,(\tfrac{1}{2}h)$, $E\,\{\tfrac{1}{2}\,(l-1)\}$ denote the greatest integers contained in $\tfrac{1}{2}h$, $\tfrac{1}{2}\,(l-1)$ respectively. The number of invariant-factors to base r is independent of A and equal to the *defect** of B; the number corresponding to $r = \infty$ is independent of B and equal to the *defect* of A.

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* If a form S be brought to the sum of p positive and q negative squares, the numbers p, q are invariants of the form (Sylvester and Jacobi); Loewy calls the smaller of p, q the *characteristic* of S, $(p+q)$ the *rank*, $(p-q)$ the *signature*, $n-(p+q)$ the *defect*.

An equivalent statement is the following (employing the notation above):—

Let σ be the absolute value of the *signature* of B; then.

$$\sigma \leqq (\text{the number of odd } h\text{'s}) + (\text{the number of even } l\text{'s}).$$

Herr Loewy gives two special cases of this, in his theorems (*a*), (*b*), § 2.]

Thursday, May 10th, 1900.

Prof. ELLIOTT, F.R.S., Vice-President, in the Chair.

Twenty-two members present.

SPECIAL GENERAL MEETING.

The Chairman, having read the By-laws bearing upon the subject of the meeting, announced that it was proposed "that By-law IV., 1 (By-laws, p. xiii.) be amended by substituting the words "half-past 5 o'clock in the afternoon" for "8 o'clock in the evening." The motion having been seconded by Dr. Larmor, and there being no other action taken, the Chairman declared the motion to be carried unanimously.

ORDINARY GENERAL MEETING.

Dr. Glaisher communicated "A Congruence Theorem relating to Eulerian Numbers and other Coefficients."

Prof. Lamb spoke briefly "On a Peculiarity of the Wave System due to the Free Vibrations of a Nucleus in an Extended Medium."

Prof. Love gave a description of Mr. Michell's "Diagrams illustrating Distributions of Stress in Two Dimensions" (a paper communicated at the January meeting).*

The following papers were taken as read:—

- The Differential Equation whose Solution is the Ratio of Two Solutions of a Linear Differential Equation: M. W. J. Fry.
- Note on the Quinquisectional Equation: Prof. L. J. Rogers.
- On the Differentiation of Single Theta-Functions: Rev. M. M. U. Wilkinson.
- Linear Substitutions Commutative with a Given Substitution: Dr. L. E. Dickson.

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* *Cf. supra*, pp. 53–61.

Lt.-Col. Cunningham showed that numbers which are expressible in the *two* different forms $N = (\mu x^2 + \nu y^2)/a = (\mu' x'^2 + \nu' y'^2)/a'$ are *usually composite*, when $\mu\nu = \mu'\nu'$; and showed how to reduce them to the forms $N = X^2 + \mu\nu Y^2 = X'^2 + \mu\nu Y'^2$, the factorization of which is known from Euler's researches.

The following presents were made to the Library :—

"Educational Times," May, 1900.

"Indian Engineering," Vol. xxvii., Nos. 11–15, March 17–April 14, 1900.

"Queen's College, Galway, Calendar for 1899–1900" ; Dublin, 1900.

"Matematisch-Naturwissenschaftliche Mitteilungen," Serie 2, Bd. ii., Heft 2 ; tuttgart, 1900.

"Supplemento al Periodico di Matematica," Anno iii., Fasc. 6, 7 ; Livorno, 1900.

"Wiadomosci Matematyczne," Tom iv., Zeszyt 1–3 ; Warsaw, 1900.

"Annals of Mathematics," Series 2, Vol. i., No. 3 ; 1900.

The following exchanges were received :—

"Proceedings of the Royal Society," Vol. lxvi., No. 428 ; 1900.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxiv., St. 4 ; Leipzig, 1900.

"Rendiconti del Circolo Matematico di Palermo," Tomo xiv., Fasc. 1, 2 ; "Annuario, 1900."

"Bulletin de la Société Mathématique de France," Tome xxviii., Fasc. 1 ; Paris, 1900.

"Bulletin of the American Mathematical Society," Series 2, Vol. vi., No. 7, April, 1900 ; New York.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. xiv., No. 1 ; Coimbra, 1900.

"Atti del R. Istituto Veneto," Tomo lvi., Disp. 8–10 ; Tomo lvii., Atti (345 pp.) ; Tomo lviii., Disp. 1–5 ; Tomo lix., Disp. 1, 2 : Supplemento al Tomo lvii. ; 1897–1900.

"Monatshefte für Mathematik und Physik," Jahrgang xi., No. 2 ; Wien, 1900.

"Bulletin des Sciences Mathématiques," Tome xxiv., Fev., 1900 ; Paris.

"Annali di Matematica," Tomo iv., Fasc. 1, 2 ; Milano, 1900.

"Archives Néerlandaises," Série 2, Tome iii., Livr. 3, 4 ; La Haye, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. ix., Fasc. 6, 7 ; Roma, 1900.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," No. 1, Bd. lii. ; 1900.

"Nyt Tidsskrift for Matematik," A. Aargang xi., Nos. 3, 4 ; Copenhagen, 1900.

"Journal of the Institute of Actuaries," Vol. xxxv., Pt. 3 ; April, 1900.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Vol. xliv., *Hefte 3, 4 ; 1900.*

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. xliv., Pts. 2, 3; 1900.

"Proceedings of the Royal Irish Academy," Series 3, Vol. v., No. 4; Dublin, 1900.

Linear Substitutions Commutative with a Given Substitution. *By* L. E. Dickson, Ph.D. Received May 8th, 1900. Communicated May 10th, 1900.

1. The object of this note is to determine the explicit form of all m-ary linear homogeneous substitutions T with coefficients in the $GF[p^n]$ which are commutative with a particular one S. For the case $n = 1$, the number of such substitutions T has been determined by M. Jordan,[*] whose method of proof was, however, limited to the consideration of a particular example. By the use of convenient notations, we may treat the general case with equal ease and, moreover, avoid the separation of the proof into two successive stages. Following M. Jordan, I first give to S its canonical form[†] S_1.

2. Let the characteristic determinant of S be

$$\Delta(K) \equiv [F_k(K)]^\alpha [F_l(K)]^\beta \dots \quad (m \equiv k\alpha + l\beta + \dots),$$

where $F_k(K)$, $F_l(K)$, ... are distinct polynomials belonging to, and irreducible in, the $GF[p^n]$. We may exhibit the roots of $F_k(K) = 0$ and of $F_l(L) = 0$ in the following notation:—

$$K_0, \quad K_1 \equiv K_0^{p^n}, \quad \dots, \quad K_{k-1} \equiv K_0^{p^{n(k-1)}};$$

$$L_0, \quad L_1 \equiv L_0^{p^n}, \quad \dots, \quad L_{l-1} \equiv L_0^{p^{n(l-1)}}.$$

To simplify the formulæ, we suppose that F_k and F_l are the only irreducible factors of $\Delta(K)$. The method is, however, seen to be general.

Corresponding to each partition of α and β into positive integers, we obtain a canonical form of an m-ary substitution in the $GF[p^n]$. Let

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[*] *Traité des Substitutions*, pp. 128–136.

[†] "Canonical Form of a Linear Homogeneous Substitution in a Galois Field," *American Journal of Mathematics*, Vol. xxii., No. 2, April, 1900. The proof of the generalization of Jordan's theorem is there made by induction.

$$\alpha \equiv a_1+a_2+\dots+a_{r+1}, \quad \beta \equiv b_1+b_2+\dots+b_{s+1}.$$

It will be convenient to let e denote any one of the integers

$$(e) \qquad 1, \; a_1+1, \; a_1+a_2+1, \; \dots, \; a_1+a_2+\dots+a_r+1;$$

and E any one of the remaining integers $\overline{\overline{<}}\, \alpha$. Let b denote any one of the integers $1, b_1+1, \dots, b_1+b_2+\dots+b_s+1$; and B any one of the remaining integers $\overline{\overline{<}}\, \beta$. The general canonical form may now be written:

$$S_1: \begin{cases} \eta'_{ij} = K_i\eta_{ij} & (i=0,1,\dots,k-1;\, j \text{ any } e) \\ \eta'_{ij} = K_i\eta_{ij}+K_i\eta_{ij-1} & (i=0,1,\dots,k-1;\, j \text{ any } E) \\ \zeta'_{ij} = L_i\zeta_{ij} & (i=0,1,\dots,l-1;\, j \text{ any } b) \\ \zeta'_{ij} = L_i\zeta_{ij}+L_i\zeta_{ij-1} & (i=0,1,\dots,l-1;\, j \text{ any } B). \end{cases}$$

3. An arbitrary linear homogeneous substitution on these indices may be exhibited as follows:

$$(T_1): \begin{cases} \eta'_{ij} = \Sigma a^{ij}_{tu}\eta_{tu}+\Sigma\beta^{ij}_{vw}\zeta_{vw} & (i=0,\dots,k-1;\, j=1,\dots,\alpha) \\ \zeta'_{ij} = \Sigma\gamma^{ij}_{tu}\eta_{tu}+\Sigma\delta^{ij}_{vw}\zeta_{vw} & (i=0,\dots,l-1;\, j=1,\dots,\beta), \end{cases}$$

where, as henceforth, the summation indices t, u, v, w run through the series

$$t=0,1,\dots,k-1; \quad v=0,1,\dots,l-1; \quad u=1,\dots,\alpha; \quad w=1,\dots,\beta.$$

We investigate the conditions under which T_1 is commutative with S_1. Equating the functions by which T_1S_1 and S_1T_1 replace η_{ie}, we get

$$K_i\sum_{t,u} a^{ie}_{tu}\eta_{tu}+K_i\sum_{v,w}\beta^{ie}_{vw}\zeta_{vw} \equiv \sum_{t,u} a^{ie}_{tu}K_t\eta_{tu}+\sum_{t,E} a^{ie}_{tE}K_t\eta_{t\,E-1}$$
$$+\sum_{v,w}\beta^{ie}_{vw}L_v\zeta_{vw}+\sum_{v,B}\beta^{ie}_{vB}L_v\zeta_{v\,B-1}.$$

This identity in the variables η and ζ requires

$$\begin{aligned} K_i a^{ie}_{tu} &= K_t a^{ie}_{tu} && (u \neq E-1), \\ K_i a^{ie}_{t\,E-1} &= K_t a^{ie}_{t\,E-1}+K_t a^{ie}_{tE}, \\ K_i\beta^{ie}_{vw} &= L_v\beta^{ie}_{vw} && (w \neq B-1), \\ K_i\beta^{ie}_{v\,B-1} &= L_v\beta^{ie}_{v\,B-1}+L_v\beta^{ie}_{vB}. \end{aligned}$$

For $t=i$, the first two equations give merely $a^{ie}_{iE}=0$. For $t \neq i$,

$K_i \neq K_t$, and the first equation gives $a^{ie}_{t\,e'-1} = 0$, e' being any integer > 1 of the set (e). If $e'-1$ is an E, the second equation gives $a^{ie}_{t\,e'-2} = 0$; in the contrary case, $e'-2 \neq E-1$, and the same result follows from the first equation. Next, according as $e'-2$ is or is not an E, the second or first equation gives $a^{ie}_{t\,e'-3} = 0$. Proceeding thus, we find that every $a^{ie}_{tc} = 0$ $(t \neq i,\ c$ arbitrary$)$.

Since $K_i \neq L_v$, the third and fourth equations require, for similar reasons, that every $\beta^{ie}_{vc} = 0$ (c arbitrary).

It follows that T_1 replaces η_{ie} by $\sum\limits_{e'} a^{ie}_{ie'}\eta_{ie'}$.

Denote by e_1 (or by e_1') an arbitrary e such that e_1+1 is an E, and by $\bar{e}_1$ any one of the remaining e's, so that $\bar{e}_1+1$ is an e. Equating the functions by which T_1S_1 and S_1T_1 replace $\eta_{i\,e_1+1}$, we get

$$K_i \sum_{t,u} a^{i\,e_1+1}_{tu}\eta_{tu} + K_i \sum_{e} a^{ie_1}_{ie}\eta_{ie} + K_i \sum_{v,w} \beta^{i\,e_1+1}_{vw}\zeta_{vw}$$
$$\equiv \sum_{t,u} a^{i\,e_1+1}_{tu} K_t\eta_{tu} + \sum_{t,E} a^{i\,e_1+1}_{tE} K_t \eta_{t\,E-1} + \sum_{v,w} \beta^{i\,e_1+1}_{vw} L_v\zeta_{vw} + \sum_{v,B} \beta^{i\,e_1+1}_{vB} L_v \zeta_{v\,B-1}.$$

Equating the coefficients of the ζ's, we find, as above, that every $\beta^{i\,e_1+1}_{vc} = 0$ (c arbitrary). In the second sum of the second member, E extends over every $E = e_1'+1$ and every E' not an $e+1$. Hence

$$a^{ie_1}_{i\bar{e}_1} = 0, \quad a^{ie_1}_{ie_1'} = a^{i\,e_1+1}_{i\,e_1'+1}, \quad a^{i\,e_1+1}_{iE'} = 0 \quad (E' \neq e+1),$$
$$a^{i\,e_1+1}_{tu} = 0 \qquad (t \neq i,\ u \neq E-1),$$
$$K_i a^{i\,e_1+1}_{t\,E-1} = K_t a^{i\,e_1+1}_{t\,E-1} + K_t a^{i\,e_1+1}_{tE} \qquad (t \neq i).$$

Applying the above argument, the last two equations give

$$a^{i\,e_1+1}_{tc} = 0 \quad (t \neq i,\ c = 1, \ldots, a).$$

Hence T_1 affects η_{ie} and $\eta_{t\,e+1}$ as follows :—

$$\eta'_{ie_1} = \sum_{e_1'} a^{ie_1}_{ie_1'}\eta_{ie_1'}, \quad \eta'_{i\bar{e}_1} = \sum_{e} a^{i\bar{e}_1}_{ie}\eta_{ie},$$
$$\eta'_{i\,e_1+1} = \sum_{e} a^{i\,e_1+1}_{ie}\eta_{ie} + \sum_{e_1'} a^{ie_1}_{ie_1'}\eta_{i\,e_1'+1}.$$

Denote by e_2 (or by e_2') an arbitrary e_1 such that e_2+2 is an E, and by $\bar{e}_2$ any one of the remaining e_1's so that every $\bar{e}_2+2$ is an e. Equating the functions by which T_1S_1 and S_1T_1 replace $\eta_{i\,e_2+2}$, we get

$$K_i \sum_{t,u} a^{i\,e_2+2}_{tu}\eta_{tu} + K_i \sum_{v,w} \beta^{i\,e_2+2}_{vw}\zeta_{vw} + \sum_{e} a^{i\,e_2+1}_{ie}\eta_{ie} + \sum_{e_1} a^{ie_2}_{ie_1}\eta_{i\,e_1+1}$$
$$= \sum a^{i\,e_2+2}_{tu} K_t\eta_{tu} + \sum a^{i\,e_2+2}_{tE} K_t\eta_{t\,E-1} + \sum \beta^{i\,e_2+2}_{vw} L_v\zeta_{vw} + \sum \beta^{i\,e_2+2}_{vB} L_v\zeta_{v\,B-1}$$

Equating the coefficients of the ζ's, we find, as above, that

$$\beta_{vc}^{i\,e_2+2} = 0 \quad (v = 0, 1, \ldots, l-1;\ c = 1, \ldots, \beta).$$

Equating the coefficients of η_{tu} $(t \neq i)$, we find, as formerly, that every

$$a_{tu}^{i\,e_2+2} = 0 \quad (t \neq i).$$

For $t = i$, we note that

$$\sum_E a_{iE}^{i\,e_2+2}\,\eta_{i\,E-1} \equiv \sum_{e_1} a_{i\,e_1+1}^{i\,e_2+2}\,\eta_{ie_1} + \sum_{e'_2} a_{i\,e'_2+2}^{i\,e_2+2}\,\eta_{i\,e'_2+1} + \sum_{E'} a_{iE'}^{i\,e_2+2}\,\eta_{i\,E'-1},$$

where E' runs over the series of E's not of the forms e_1+1 or $e+2$. But an $\bar{e}_1+2 \equiv (\bar{e}_1+1)+1$ is an $e+1$, and an $\bar{e}_2+2$ is not an E. Hence E' extends over those integers $\overset{=}{<} u$ which are of none of the forms e, e_1+1, e_2+2, all three of which are distinct. Hence every

$$a_{ie_1}^{i\,e_2+1} = a_{i\,e_1+1}^{i\,e_2+2}, \quad a_{i\bar{e}_1}^{i\,e_2+1} = 0, \quad a_{i\bar{e}_2}^{i\,e_2} = 0,$$

$$a_{ie'_2}^{i\,e_2} = a_{i\,e'_2+2}^{i\,e_2+2}, \quad a_{iE'}^{i\,e_2+2} = 0.$$

Hence T_1 affects the indices η_{ie}, $\eta_{i\,e+1}$, $\eta_{i\,e+2}$ as follows:—

$$\eta'_{i\bar{e}_1} = \sum_e a_{ie}^{i\bar{e}_1}\,\eta_{ie},$$

$$\begin{cases} \eta'_{i\bar{e}_2} = \sum_{e_1} a_{ie_1}^{i\bar{e}_2}\,\eta_{ie_1}, \\ \eta'_{i\,\bar{e}_2+1} = \sum_e a_{ie}^{i\,e_2+1}\,\eta_{ie} + \sum_{e_1} a_{ie_1}^{i\,e_2}\,\eta_{i\,e_1-1}, \end{cases}$$

$$\begin{cases} \eta'_{ie_2} = \sum_{e'_2} a_{ie'_2}^{i\,e_2}\,\eta_{ie'_2}, \\ \eta'_{i\,e_2+1} = \sum_{e_1} a_{ie_1}^{i\,e_2+1}\,\eta_{ie_1} + \sum_{e'_2} a_{ie'_2}^{i\,e_2}\,\eta_{i\,e'_2+1}, \\ \eta'_{i\,e_2+2} = \sum_e a_{ie}^{i\,e_2+2}\,\eta_{ie} + \sum_{e_1} a_{ie_1}^{i\,e_2+1}\,\eta_{i\,e_1+1} + \sum_{e'_2} a_{ie'_2}^{i\,e_2}\,\eta_{i\,e'_2+2}. \end{cases}$$

Proceeding as before, we separate the e_2 into the categories e_3 and $\bar{e}_3$, such that every e_3+3 is an E and every $\bar{e}_3+3$ is an e. We find that no simplification takes place in $\eta'_{i\bar{e}_1}$, $\eta'_{i\bar{e}_2}$, $\eta'_{i\,\bar{e}_2+1}$, nor in η'_{ie_2}, $\eta'_{i\,e_2+1}$, $\eta'_{i\,e_2+2}$, when e_2 is an $\bar{e}_3$. Simplifications arise when e_2 is an e_3, viz.:

$$\begin{cases} \eta'_{ie_3} = \sum_{e'_3} a_{ie'_3}^{i\,e_3}\,\eta_{ie'_3}, \\ \eta'_{i\,e_3+1} = \sum_{e_3} a_{ie_3}^{i\,e_3+1}\,\eta_{ie_3} + \sum_{e'_3} a_{ie'_3}^{i\,e_3}\,\eta_{i\,e'_3+1}, \\ \eta'_{i\,e_3+2} = \sum_{e_1} a_{ie_1}^{i\,e_3+2}\,\eta_{ie_1} + \sum_{e_2} a_{ie_2}^{i\,e_3+1}\,\eta_{i\,e_2+1} + \sum_{e'_3} a_{ie'_3}^{i\,e_3}\,\eta_{i\,e'_3+2}, \\ \eta'_{i\,e_3+3} = \sum_e a_{ie}^{i\,e_3+3}\,\eta_{ie} + \sum_{e_1} a_{ie_1}^{i\,e_3+2}\,\eta_{i\,e_1+1} + \sum_{e_2} a_{ie_2}^{i\,e_3+1}\,\eta_{i\,e_2+2} + \sum_{e'_3} a_{ie'_3}^{i\,e_3}\,\eta_{i\,e'_3+3}. \end{cases}$$

The law of the formation of the η'_{ij} is now evident, and may be verified by simple induction. In particular, T_1 replaces each η_{ij} by a function of the η_{iu} alone. Similarly, T_1 replaces each ζ_{ij} by a function of the ζ_{iw} only.

4. Consider, as an example, a substitution S_1 which involves only the indices η_{ij}, and for which $a_1 = 3$, $a_2 = 3$, $a_3 = 2$. Then

$$e = 1, 4, 7\,; \quad E = 2, 3, 5, 6, 8\,;$$

$$e_1 = 1, 4, 7\,; \quad \text{no } \bar{e}_1\,; \quad e_2 = 1, 4\,; \quad \bar{e}_2 = 7\,; \quad \text{no } e_3.$$

The most general substitution T_1 commutative with S_1 has the form

$$\begin{array}{|cc|ccc|ccc|cc|}
\hline
 & & \eta_{i1} & \eta_{i4} & \eta_{i7} & \eta_{i2} & \eta_{i5} & \eta_{i8} & \eta_{i3} & \eta_{i6} \\
\hline
\eta'_{i7} & = & a^{i7}_{i1} & a^{i7}_{i4} & a^{i7}_{i7} & & & & & \\
\eta'_{i8} & = & a^{i8}_{i1} & a^{i8}_{i4} & a^{i8}_{i7} & a^{i7}_{i1} & a^{i7}_{i4} & a^{i7}_{i7} & & \\
\hline
\eta'_{i1} & = & a^{i1}_{i1} & a^{i1}_{i4} & & & & & & \\
\eta'_{i2} & = & a^{i2}_{i1} & a^{i2}_{i4} & a^{i2}_{i7} & a^{i1}_{i1} & a^{i1}_{i4} & & & \\
\eta'_{i3} & = & a^{i3}_{i1} & a^{i3}_{i4} & a^{i3}_{i7} & a^{i2}_{i1} & a^{i2}_{i4} & a^{i2}_{i7} & a^{i1}_{i1} & a^{i1}_{i4} \\
\hline
\eta'_{i4} & = & a^{i4}_{i1} & a^{i4}_{i4} & & & & & & \\
\eta'_{i5} & = & a^{i5}_{i1} & a^{i5}_{i4} & a^{i5}_{i7} & a^{i4}_{i1} & a^{i4}_{i4} & & & \\
\eta'_{i6} & = & a^{i6}_{i1} & a^{i6}_{i4} & a^{i6}_{i7} & a^{i5}_{i1} & a^{i5}_{i4} & a^{i5}_{i7} & a^{i4}_{i1} & a^{i4}_{i4} \\
\hline
\end{array}$$

holding for $i = 0, 1, \ldots, k-1$. By inspection, its determinant equals

$$(a^{i7}_{i7})^3 \begin{vmatrix} a^{i1}_{i1} & a^{i1}_{i4} \\ a^{i4}_{i1} & a^{i4}_{i4} \end{vmatrix}^3.$$

5. The indices $\eta_{i1}, \ldots, \eta_{ia}$ are linear functions of the initial indices $\xi_1, \ldots, \xi_m$, having as coefficients polynomials in K_i. Likewise, $\zeta_{i1}, \ldots, \zeta_{i\beta}$ are linear functions of $\xi_1, \ldots, \xi_m$ involving L_i. Let us return from the indices η_{ij}, ζ_{ij} to the initial indices ξ_i. By hypothesis, S_1 becomes S, a substitution having its coefficients in the $GF[p^n]$. Let T_1

become T. Under what conditions will T have its coefficients in the same field? Remembering that T_1 replaces η_{ij}, ζ_{ij} by functions of the respective forms

$$\sum_{u=1}^{\alpha} a_{iu}^{ij}\eta_{iu}, \quad \sum_{w=1}^{\beta} \delta_{iw}^{ij}\zeta_{iw},$$

it is evidently necessary and sufficient that a_{iu}^{ij} be the same function of K_i for $i = 1, \ldots, k-1$ that a_{0u}^{0j} is of K_0, and that δ_{iw}^{ij} be the same function of L_i for $i = 1, \ldots, l-1$ that δ_{0w}^{0j} is of L_0. Expressed otherwise, these conditions are

$$a_{iu}^{ij} = (a_{0u}^{0j})^{p^{ni}}, \quad \delta_{iw}^{ij} = (\delta_{0w}^{0j})^{p^{ni}}.$$

Hence T_1 is completely determined from the functions by which it replaces η_{0j} $(j = 1, \ldots, \alpha)$ and ζ_{0j} $(j = 1, \ldots, \beta)$. The final theorem is as follows:—

To determine the most general m-ary linear homogeneous substitution T with coefficients in the GF $[p^n]$ *which is commutative with a particular one S, we give to S its canonical form* S_1, *which may be expressed as a product,*

$$S_1 \equiv y_0 y_1 \ldots y_{k-1} z_0 z_1 \ldots z_{l-1} \ldots,$$

y_i, z_i *denoting the respective substitutions—*

$$y_i: \quad \eta_{ie} = K_i \eta_{ie}, \quad \eta_{iE} = K_i(\eta_{iE} + \eta_{iE-1}),$$

$$z_i: \quad \zeta_{ib} = L_i \zeta_{ib}, \quad \zeta_{iB} = L_i(\zeta_{iB} + \zeta_{iB-1}).$$

Then must T_1 *(T written in the indices* η_{ij}, ζ_{ij}*) be expressible as a product*

$$T_1 \equiv Y_0 Y_1 \ldots Y_{k-1} Z_0 Z_1 \ldots Z_{l-1} \ldots,$$

where Y_0 *affects only the indices* η_{0u}, *the coefficients being given by the law explained at the end of § 3, and where* Y_i *is obtained from* Y_0 *by raising its coefficients to the power* p^{ni}; *with similar remarks for the substitutions* Z_i.

A Congruence Theorem relating to Eulerian Numbers and other Coefficients. By J. W. L. Glaisher.

Read May 10th, 1900. Received May 28th, 1900.

1. In the *Comptes Rendus* for 1861[*] Sylvester gave without proof the theorem that, if $(p-1)p^i$ is a factor of $2n$, then, if p is a prime of the form $4k+1$, p^{i+1} will be a factor of E_n, and, if p is of the form $4k-1$, p^{i+1} will be a factor of $(-1)^{n-1}2+E_n$, where E_n is the n^{th} Eulerian number.

The principal object of this paper is to prove two general theorems, from either of which Sylvester's results in the case $i=0$ may be deduced, as well as corresponding results relating to the I-numbers and other coefficients.

In the latter portion of the paper (§§ 39–65) these general theorems are applied to obtain the residues, mod p, of various systems of series containing $\frac{1}{2}(p-3)$ terms.

I. *Residues of the Bernoullian Functions with Uneven Suffixes, and Applications* (§§ 2–38).

2. Let p be any uneven prime, and let A_r denote the sum of the products of the numbers $1, 2, \ldots, p-1$ taken r together. Then we have evidently

$$x^{p-1}+A_1x^{p-2}+A_2x^{p-3}+\ldots+A_{p-2}x+A_{p-1} = (x+1)(x+2)\ldots(x+p-1).$$

In a recent paper in the *Quarterly Journal*[†] it has been shown that, if r is even and $<p-1$, A_r is divisible by p, and that, if r is uneven and >1, A_r is divisible by p^2, and that the residues of the quotients are

$$\frac{A_{2t}}{p} \equiv (-1)^t\frac{B_t}{2t}, \text{ mod } p \ \left(t<\frac{p-1}{2}\right),$$

$$\frac{A_{2t+1}}{p^2} \equiv (-1)^{t+1}\frac{(2t+1)B_t}{4t}, \text{ mod } p \ (t>0).$$

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[*] Vol. LII., p. 163.

[†] "On the Residues of the Sums of Products of the first $p-1$ Numbers and their Products to Modulus p^2 or p^3," Vol. XXXI., pp. 321–353. The formulæ quoted occur on pp. 326, 327.

It was also shown that

$$\frac{A_{p-1}+1}{p} \equiv J, \bmod p,$$

where B_t is the t^{th} Bernoullian number and

$$J = -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}.$$

The value of A_1 is $\frac{1}{2}p(p-1)$, so that it is divisible by p only.

3. Multiplying the equation in § 2 by x and dividing by p, we therefore find

$$\frac{x^p}{p} - \tfrac{1}{2}x^{p-1} - \frac{B_1}{2}x^{p-2} + \frac{B_2}{4}x^{p-4} - \ldots + (-1)^{\frac{1}{2}(p-3)} \frac{B_{\frac{1}{2}(p-3)}}{p-3} x^3 + \frac{A_{p-1}}{p} x$$

$$\equiv \frac{x(x+1)\ldots(x+p-1)}{p}, \bmod p.$$

Now the Bernoullian function $B_n(x)$, n being uneven, is defined by the equation

$$B_n(x) = \frac{x^n}{n} - \tfrac{1}{2}x^{n-1} + \frac{n-1}{2!} B_1 x^{n-2} - \frac{(n-1)(n-2)(n-3)}{4!} B_2 x^{n-4} + \ldots$$

$$\ldots + (-1)^{\frac{1}{2}(n+1)} B_{\frac{1}{2}(n-1)} x,$$

and therefore

$$B_p(x) \equiv \frac{x^p}{p} - \tfrac{1}{2}x^{p-1} - \frac{B_1}{2}x^{p-2} + \frac{B_2}{4}x^{p-4} - \ldots + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} x, \bmod p.$$

Substituting in the preceding congruence, we find

$$B_p(x) + (-1)^{\frac{1}{2}(p-1)} B_{\frac{1}{2}(p-1)} x + \frac{A_{p-1}}{p} x \equiv \frac{x(x+1)\ldots(x+p-1)}{p}, \bmod p.$$

Now

$$(-1)^{\frac{1}{2}(p-1)} B_{\frac{1}{2}(p-1)} + \frac{A_{p-1}}{p}$$

$$\equiv (-1)^{\frac{1}{2}(p-1)} B_{\frac{1}{2}(p-1)} + J - \frac{1}{p}, \bmod p,$$

$$\equiv (-1)^{\frac{1}{2}(p-1)} B_{\frac{1}{2}(p-1)} - 1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)}, \bmod p,$$

$$\equiv -1, \bmod p;$$

so that the congruence becomes

$$B_p(x) - x \equiv \frac{x(x+1)\ldots(x+p-1)}{p}, \bmod p.$$

4. When x is a positive integer prime to p, one of the p consecutive numbers $x, x+1, \ldots, x+p-1$ must be divisible by p, and the other $p-1$ numbers must have residues $1, 2, 3, \ldots, p-1$ with respect to p.

If therefore $x = kp+t$, t being $<p$, the formula gives

$$B_p(x)-x \equiv -(k+1), \quad \text{mod } p,$$

and in particular, if x be any number $<p$,

$$B_p(x) \equiv x-1, \quad \text{mod } p.$$

5. These formulæ can be readily verified, for, when x is a positive integer,

$$B_p(x) = 1^{p-1}+2^{p-1}+3^{p-1}+\ldots+(x-1)^{p-1};$$

and therefore, since $r^{p-1} \equiv 1$, mod p, unless r is a multiple of p, we have, if $x<p$,

$$B_p(x) \equiv x-1, \quad \text{mod } p,$$

and, if $x = kp+t$,

$$B_p(x) \equiv x-1-k, \quad \text{mod } p,$$

since the k terms $p^{p-1}, (2p)^{p-1}, \ldots, (kp)^{p-1}$ are $\equiv 0$, not 1, mod p.

6. Now let $x = \dfrac{1}{r}$, where r is a positive integer prime to p.

The general formula

$$B_p(x) \equiv x+\frac{x(x+1)\ldots(x+p-1)}{p}, \quad \text{mod } p,$$

then becomes

$$r^p B_p\left(\frac{1}{r}\right) \equiv r^{p-1}+\frac{(r+1)(2r+1)\ldots\{(p-1)r+1\}}{p}, \quad \text{mod } p,$$

whence $r^p B_p\left(\dfrac{1}{r}\right) \equiv 1+\dfrac{(r+1)(2r+1)\ldots\{(p-1)r+1\}}{p}$, mod p.

7. Since r is prime to p, the numbers $r, 2r, \ldots, (p-1)r$ have to mod p the system of residues $1, 2, \ldots, p-1$; and therefore the numbers $r+1, 2r+1, \ldots, (p-1)r+1$ have the system of residues $2, 3, \ldots, p-1, p$; that is to say, one of the numbers $r+1, 2r+1, \ldots, (p-1)r+1$ is divisible by p, and the other numbers give the residues $2, 3, \ldots, p-1$.

To determine which is the factor divisible by p, we notice that, if

$\lambda r+1$ is this factor, we must have $\lambda r+1 = qp$, where $q<r$ and $\equiv \frac{1}{p}$, mod r. The product of the factors 2, 3, ..., $p-1 \equiv -1$, mod p, and therefore we have the formula

$$r^p B_p\left(\frac{1}{r}\right) \equiv 1-q, \quad \text{mod } p,$$

where q is the least positive residue of $\frac{1}{p}$, mod r.

8. We may conveniently express the least positive residue of any quantity a, mod r, by $[a]_r$.

Using this notation, the result just obtained may be written

$$r^p B_p\left(\frac{1}{r}\right) \equiv 1-\left[\frac{1}{p}\right]_r, \quad \text{mod } p.$$

9. This formula enables us to assign the residue of $r^p B_p\left(\frac{1}{r}\right)$, mod p, for all values of r and all (prime) values of p.

In general, if $p = kr+s$, where $s<r$ and is necessarily prime to r, since p is prime,

$$\left[\frac{1}{p}\right]_r = \left[\frac{1}{s}\right]_r.$$

Thus the number of residues of $r^p B_p\left(\frac{1}{r}\right)$ is equal to the number of admissible values of s, that is, to the number of numbers less than r and prime to it.

10. The residue of $r^p B_p\left(\frac{l}{r}\right)$ may be expressed in an exactly similar manner; for, putting $x = \frac{l}{r}$ in § 6, l being prime to r, we have

$$r^p B_p\left(\frac{l}{r}\right) \equiv lr^{p-1} + \frac{l(r+l)(2r+l)\dots\{(p-1)r+l\}}{p}, \quad \text{mod } p,$$

$$\equiv l + \frac{l(r+l)(2r+l)\dots\{(p-1)r+l\}}{p}, \quad \text{mod } p.$$

Now, of the p numbers l, $r+l$, $2r+l$, ... $\{(p-1)r+l\}$, one, say $\lambda r+l$, is divisible by p, and the others $\equiv 2, 3, \dots, p-1$, mod p, so that we find

$$r^p B_p\left(\frac{l}{r}\right) \equiv l-q_l, \quad \text{mod } p,$$

where q_l is the least root of the congruence $px \equiv l$, mod r; or, using the notation of § 8,

$$r^p B_p\left(\frac{l}{r}\right) \equiv l - \left[\frac{l}{p}\right]_r, \quad \text{mod } p.\text{*}$$

11. It is known that

$$B_n \equiv B_{n-k(p-1)}(x), \quad \text{mod } p,\text{†}$$

and therefore
$$r^{k(p-1)+p} B_{k(p-1)+p}(x) \equiv r^p B_p(x), \quad \text{mod } p.$$

Putting $x = \dfrac{l}{r}$, and replacing k by $k-1$ in this formula, we have

$$r^{k(p-1)+1} B_{k(p-1)+1}\left(\frac{l}{r}\right) \equiv r^p B_p\left(\frac{l}{r}\right) \equiv l - \left[\frac{l}{p}\right]_r, \quad \text{mod } p.$$

It has thus been shown that, if p be an uneven prime, and $p-1$ be a divisor of $2n$, then, if $l < p+r$,

$$r^{2n+1} B_{2n+1}\left(\frac{l}{r}\right) \equiv l - \left[\frac{l}{p}\right]_r, \quad \text{mod } p.$$

If $l \geqq p+r$, the formula requires modification in the manner indicated in the note to the last section.

12. As special values of $r^{2n+1} B_{2n+1}\left(\dfrac{1}{r}\right)$, we have‡

$$4^{2n+1} B_{2n+1}(\tfrac{1}{4}) = (-1)^{n+1} E_n,$$

$$3^{2n+1} B_{2n+1}(\tfrac{1}{3}) = (-1)^{n+1} I_n,$$

$$6^{2n+1} B_{2n+1}(\tfrac{1}{6}) = (-1)^{n+1} J_n,$$

---

* This congruence holds good so long as $l < p+r$, but for greater values of l it requires modification, q_l not being then, necessarily, the least root of the congruence $px \equiv l$, mod r. In general it can be shown that, if $l = kp+t$, where $t > 0$ and $< p$, then

$$r^p B_p\left(\frac{l}{r}\right) \equiv l - q_l, \text{ mod } p,$$

where q_l is not x_1, the least root of the congruence $px \equiv l$, mod r, but $x_1 + mr$, mr being such a multiple of r as will bring $x_1 + mr$ within the range of numbers $k+1$, $k+2, \ldots, k+r$, *i.e.*, q_l is that root of the congruence $px \equiv l$, mod r, which lies between $k+1$ and $k+r$ both inclusive. The value of q_l may be expressed in terms of k and t by $k + \left[\dfrac{t}{p}\right]_r$.

In most of the applications of the theorem l is, or can be so chosen as to be, $< p+r$; the principal exceptions in this paper occur in §§ 33–38 and 60–65.

[The residue of $\dfrac{l(r \div l)(2r+l)\ldots\{(p-1)r+l\}}{p}$, mod p, that is, the value of q_l, forms the subject of the first part of a paper "Residue of the Product of p Numbers in Arithmetical Progression, mod p^2 and p^3" (*Messenger of Mathematics*, Vol. XXX., pp. 71–92), which has been written since this paper was communicated to the Society.]

† *Proc. Lond. Math. Soc.*, Vol. XXXI., p. 206.

‡ *Quarterly Journal*, Vol. XXIX., pp. 31, 35, 44, or *Messenger*, Vol. xxvi., p. 179.

where E_n is the n^{th} Eulerian number, and I_n and J_n are the numbers so denoted in Vol. XXXI., pp. 216, 228 of the *Proceedings*, viz., E_n, I_n, J_n are the coefficients in the expansions

$$\frac{1}{\cos x} = E_0 + \frac{E_1}{2!}x^2 + \frac{E_2}{4!}x^4 + \frac{E_3}{6!}x^6 + \&c.,$$

$$\frac{1}{2\cos x+1} = \tfrac{2}{3}\left\{I_0 + \frac{I_1}{2!}x^2 + \frac{I_2}{4!}x^4 + \frac{I_3}{6!}x^6 + \&c.\right\},$$

$$\frac{2\cos x}{2\cos 2x+1} = \tfrac{1}{3}\left\{J_0 + \frac{J_1}{2!}x^2 + \frac{J_2}{4!}x^4 + \frac{J_3}{6!}x^6 + \&c.\right\}.$$

13. It is convenient to call the admissible values of p the *Staudt factors* for n, *i.e.*, the Staudt factors for n are those values of p for which $p-1$ is a divisor of $2n$. Thus a number p is a Staudt factor for n, if (i.) p is prime, (ii.) $p-1$ is a divisor of $2n$.*

14. Using the formula

$$r^p B_p\left(\frac{1}{r}\right) \equiv 1 - \left[\frac{1}{p}\right]_r, \quad \text{mod } p,$$

and taking the case $r = 4$, we have

$$\left[\frac{1}{p}\right]_4 = [1]_4 = 1, \text{ if } p \text{ is of the form } 4k+1,$$

and $$= [\tfrac{1}{3}]_4 = 3, \quad ,, \quad ,, \quad 4k+3.$$

Thus, p being any uneven Staudt factor for n,

$$E_n \equiv 0 \text{ or } (-1)^n\, 2, \quad \text{mod } p,$$

according as p is of the form $4k+1$ or $4k+3$.

This is the case $i = 0$ of Sylvester's theorem referred to in § 1.

15. For $r = 3$, we have

$$\left[\frac{1}{p}\right]_3 = [1]_3 = 1, \text{ if } p \text{ is of the form } 3k+1,$$

and $$= [\tfrac{1}{2}]_3 = 2, \quad ,, \quad ,, \quad 3k+2,$$

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* In previous papers (*Messenger*, Vol. XXIX., pp. 49, 129; *Quarterly Journal*, Vol. XXXI., p. 261) I have called values of p which satisfy these conditions *Staudt factors of B_n*. As the connexion is solely between the numbers p and n, it seems unnecessary to introduce B_n. The Staudt factors for n are the prime factors whose product forms the denominator of B_n.

so that, p being any uneven Staudt factor for n,

$$I_n \equiv 0 \text{ or } (-1)^n, \text{ mod } p,$$

according as p is of the form $3k+1$ or $3k+2$.

16. The quantities I_n formed the subject of a paper in the last volume of the *Proceedings.*[*] That paper contains a table of the first thirteen I's, by means of which I have verified the theorem up to $n = 13$.

The numbers 2 and 3 are Staudt factors for all values of n. The number 2 is excluded, as the modulus is always supposed to be an uneven prime. The number 3 is excluded in this case, as $r = 3$ and r must be prime to p. The residues given by the theorem with respect to the other Staudt factors are as follows:—

$I_2 \equiv 1$, mod 5,

$I_3 \equiv 0$, mod 7,

$I_4 \equiv 1$, mod 5,

$I_5 \equiv -1$, mod 11.

$I_6 \equiv 0$, mods 7, 13, $\equiv 1$, mod 5,

I_7 (no admissible Staudt factor),

$I_8 \equiv 1$, mods 5, 17,

$I_9 \equiv 0$, mods 7, 19,

$I_{10} \equiv 1$, mods 5, 11,

$I_{11} \equiv -1$, mod 23,

$I_{12} \equiv 0$, mods 7, 13, $\equiv 1$, mod 5,

I_{13} (no admissible Staudt factor).

These residues agree with those obtained from the table of I_n.

17. For $r = 6$ we have

$$\left[\frac{1}{p}\right]_6 = [1]_6 = 1, \text{ if } p \text{ is of the form } 6k+1,$$

and

$$= [\tfrac{1}{5}]_6 = 5, \quad \text{,,} \quad \text{,,} \quad 6k+5;$$

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[*] "On a Set of Coefficients analogous to the Eulerian Numbers," Vol. xxxi., pp. 216–235.

so that
$$J_n \equiv 0 \text{ or } (-1)^n 4, \text{ mod } p,$$
according as p is of the form $6k+1$ or $6k+5$.

This result may be derived from § 15 by means of the formula
$$J_n = (2^{2n+1}+2)\, I_n,$$
for, when $p-1$ is a divisor of $2n$, $2^{2n} \equiv 1$, mod p, and therefore
$$J_n \equiv 4I_n, \text{ mod } p.$$

18. The Bernoullian function $A'_n(x)$, when n is uneven, may be defined by the equation
$$A'_n(x) = B_n(x) - 2^n B_n(\tfrac{1}{2}x).^*$$
Thus we derive from § 6
$$r^p A'_p\left(\frac{1}{r}\right) \equiv \frac{(r+1)(2r+1)\dots\{(p-1)r+1\}}{p}$$
$$-\frac{(2r+1)(4r+1)\dots\{2(p-1)r+1\}}{p}, \text{ mod } p,$$
and therefore
$$r^p A'_p\left(\frac{1}{r}\right) \equiv q'-q, \text{ mod } p,$$
where q, q' are the least residues given by the congruences
$$px \equiv 1, \text{ mod } r,$$
$$px \equiv 1, \text{ mod } 2r,$$
respectively.

This result may be expressed by the formula
$$r^p A'_p\left(\frac{1}{r}\right) \equiv \left[\frac{1}{p}\right]_{2r} - \left[\frac{1}{p}\right]_r, \text{ mod } p.$$

19. More generally we have in the same manner
$$r^p A'_p\left(\frac{l}{r}\right) \equiv \frac{l(r+l)(2r+l)\dots\{(p-1)r+l\}}{p}$$
$$-\frac{l(2r+l)(4r+l)\dots\{2(p-1)r+l\}}{p}, \text{ mod } p;$$
whence we find
$$r^p A'_p\left(\frac{l}{r}\right) \equiv \left[\frac{l}{p}\right]_{2r} - \left[\frac{l}{p}\right]_r, \text{ mod } p.†$$

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* The expression for $A'_n(x)$ in powers of x was given in *Proc. Lond. Math. Soc.*, Vol. xxxi., p. 203.

† In this formula l must be $< p+r$; otherwise a modification is requisite of the same kind as that stated in the note to § 10.

20. It is not necessary that l should be prime to $2r$, but, if l be even, $= 2l'$, then

$$\frac{l(2r+l)(4r+l)\ldots\{2(p-1)r+l\}}{p}$$

$$= 2^p\frac{l'(r+l')(2r+l')\ldots\{(p-1)r+l'\}}{p} \equiv 2\left[\frac{l'}{p}\right]_r, \text{ mod } p;^*$$

so that in this case the formula may be written

$$r^pA'_p\left(\frac{l}{r}\right) \equiv 2\left[\frac{l'}{p}\right]_r - \left[\frac{l}{p}\right], \text{ mod } p.$$

21. The expression $\left[\frac{l}{p}\right]_{2r} - \left[\frac{l}{p}\right]_r$

can only have the values zero and r; for the expression is $\alpha - \beta$, where α is the least root of the congruence

$$px \equiv l, \quad \text{mod } 2r,$$

and β is the least root of the congruence

$$px \equiv l, \quad \text{mod } r.$$

Now, if $\alpha < r$, it is clear that β must $= \alpha$, and, if $\alpha > r$, we must have $\beta = \alpha - r$. Thus $\alpha - \beta$ has the value 0 or r according as the least root of the congruence $px \equiv l$, mod $2r$, is $<$ or $>$ r.†

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* This reasoning is general and shows that, if $l = ml'$, then

$$\left[\frac{l}{p}\right]_{mr} = m\left[\frac{l'}{p}\right]_r.$$

We may easily prove this formula directly, for $\left[\frac{l}{p}\right]_{mr}$ is the least value of x given by the congruence $px \equiv l$, mod mr, and, if $l = ml'$, the congruence becomes $p\frac{x}{m} \equiv l'$, mod r, from which the least value of x is $m\left[\frac{l'}{p}\right]_r$.

† If the least root of the congruence $px \equiv l$, mod $2r$, is $= r$, we must have $pr \equiv l$, mod $2r$, and therefore l must be an uneven multiple of r, $= mr$ say, m being uneven, in which case

$$\alpha - \beta = \left[\frac{mr}{p}\right]_{2r} - \left[\frac{mr}{p}\right]_r = r\left[\frac{m}{p}\right]_2 - r\left[\frac{m}{p}\right]_1 = 0.$$

22. The result obtained in § 19 therefore shows that

$$r^p A_p'\left(\frac{l}{r}\right) \equiv 0 \text{ or } r, \mod p,$$

according as the least root of the congruence

$$px \equiv l, \mod 2r,$$

is $< r$ or $> r$. The residue is zero if the root is equal to r.

23. If we put $$\left(\frac{l}{p}\right)_{2r} = \left[\frac{l}{p}\right]_{2r} - \left[\frac{l}{p}\right]_{r},$$

then the symbol $\left(\frac{l}{p}\right)_{2r}$ denotes 0 if the least root of the congruence

$$px \equiv l, \mod 2r,$$

is $< r$ or $= r$, and denotes r if the least root $> r$. We may therefore express the result of § 19 in the convenient form

$$r^p A_p'\left(\frac{l}{r}\right) \equiv \left(\frac{l}{p}\right)_{2r}, \mod p.$$

24. Since $$A_n'(x) \equiv A'_{n-k(p-1)}, \mod p,\text{*}$$

we can, by proceeding as in § 11, apply the results obtained in §§ 19 and 22 to show that, if p is any uneven prime, and $p-1$ is a divisor of $2n$ (that is, if p is an uneven Staudt factor for n), then

$$r^{2n+1} A'_{2n+1}\left(\frac{l}{r}\right) \equiv \left[\frac{l}{p}\right]_{2r} - \left[\frac{l}{p}\right]_{r}, \mod p,$$

$$\equiv \left(\frac{l}{p}\right)_{2r}, \mod p,$$

the expression on the right-hand side being zero or r according as $\left[\frac{l}{p}\right]_{2r} \leqq$ or $> r$.

If l is not $< p+r$, the formula requires modification in the manner indicated in the note to § 19.

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* *Proc. Lond. Math. Soc.*, Vol. xxxi., p. 204.

25. As special values of $r^{2n+1}A'_{2n+1}\left(\frac{1}{r}\right)$, we have*

$$2^{2n+1}A'_{2n+1}\left(\tfrac{1}{2}\right) = (-1)^n E_n,$$
$$4^{2n+1}A'_{2n+1}\left(\tfrac{1}{4}\right) = (-1)^n 2P_n,$$
$$3^{2n+1}A'_{2n+1}\left(\tfrac{1}{3}\right) = (-1)^n H_n,$$
$$6^{2n+1}A'_{2n+1}\left(\tfrac{1}{6}\right) = (-1)^n \frac{3^{2n+1}+3}{2} E_n,$$

where P_n and H_n are defined as coefficients in the expansions

$$\frac{\cos x}{\cos 2x} = P_0 + \frac{P_1}{2!}x^2 + \frac{P_2}{4!}x^4 + \frac{P_3}{6!}x^6 + \&\text{c.},$$
$$\frac{1}{2\cos x - 1} = \tfrac{2}{3}\left\{H_0 + \frac{H_1}{2!}x^2 + \frac{H_2}{4!}x^4 + \frac{H_3}{6!}x^6 + \&\text{c.}\right\}.$$

26. We thus have, p being any uneven Staudt factor for n,

$$E_n \equiv (-1)^n \left(\frac{1}{p}\right)_4, \text{ mod } p,$$
$$2P_n \equiv (-1)^n \left(\frac{1}{p}\right)_8 \quad ,,$$
$$H_n \equiv (-1)^n \left(\frac{1}{p}\right)_6 \quad ,,$$
$$\frac{3^{2n+1}+3}{2}E_n \equiv (-1)^n \left(\frac{1}{p}\right)_{12} \quad ,,$$

27. If $p = 4k+1$, $\qquad \left(\frac{1}{p}\right)_4 = 0,$

for the least root of $x \equiv 1$, mod 4, is $x = 1$, which < 2, and, if $p = 4k+3$,

$$\left(\frac{1}{p}\right)_4 = 2,$$

for the least root of $3x \equiv 1$, mod 4, is $x = 3$, which > 2.

The first congruence in § 26 therefore gives $E_n \equiv 0$ or $(-1)^n 2$, mod p, according as p is of the form $4k+1$ or $4k+3$ (§ 14).

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* *Quarterly Journal*, Vol. xxix., p. 107; and *Messenger*, Vol. xxvi., p. 178.

28. Taking now the last of the four congruences in § 26, we have

$$\left(\frac{1}{p}\right)_{12} = 0, \quad \text{if} \quad p = 12k+1 \text{ or } 12k+5,$$

and $$= 6, \quad \text{if} \quad p = 12k+7 \text{ or } 12k+11,$$

for the least roots of the congruences

$$x \equiv 1, \quad 5x \equiv 1, \quad 7x \equiv 1, \quad 11x \equiv 1, \quad \text{mod } 12,$$

are 1, 5, 7, 11 respectively, of which the first two < 6, and the last two > 6.

Thus the right-hand side of the congruence is 0 or $(-1)^n 6$ according as n is of the form $4k+1$ or $4k+3$.

This result agrees with § 14, for, if p is a Staudt factor for n,

$$\frac{3^{2n+1}+3}{2} \equiv \frac{3+3}{2} \equiv 3, \quad \text{mod } p.$$

29. The third congruence of § 26 gives

$$H_n = 0 \text{ or } (-1)^n 3, \quad \text{mod } p,$$

according as p is of the form $6k+1$ or $6k+5$.

This result agrees with § 15, for

$$H_n = (2^{2n+1}+1) I_n;$$

so that, p being any uneven Staudt factor for n,

$$H_n \equiv 3 I_n, \quad \text{mod } p.$$

30. Considering now the second congruence in § 26, we have

$$\left(\frac{1}{p}\right)_8 = 0, \quad \text{if} \quad p = 8k+1 \text{ or } 8k+3,$$

and $$= 4, \quad \text{if} \quad p = 8k+5 \text{ or } 8k+7,$$

for the least roots of the congruences

$$x \equiv 1, \quad 3x \equiv 1, \quad 5x \equiv 1, \quad 7x \equiv 1, \quad \text{mod } 8,$$

are respectively 1, 3, 5, 7.

Thus we find that

$$P_n \equiv 0 \text{ or } (-1)^n 2, \quad \text{mod } p,$$

according as p is of the forms $8k+1$ and $8k+3$, or of the forms $8k+5$ *and* $8k+7$.

31. Since 3 is a Staudt factor for all values of n, this theorem shows that all the P's must be divisible by 3. With respect to the other Staudt factors, it shows that

$$\begin{array}{ll} P_2 \equiv 2, & \text{mod } 5, \\ P_3 \equiv -2, & \text{mod } 7, \\ P_4 \equiv 2, & \text{mod } 5, \\ P_5 \equiv 0, & \text{mod } 11, \\ P_6 \equiv 2, & \text{mods } 5, 7, 13. \end{array}$$

These results I have verified. The values of the first five P's were given in the *Quarterly Journal*, Vol. XXIX., p. 63.* The value of P_6 (viz., $P_6 = 7828053417$) was calculated for the purpose of this verification.

32. The principal formulæ which have been established in this paper are

$$\text{(i.)}\quad r^p B_p\left(\frac{l}{r}\right) \equiv l - \left[\frac{l}{p}\right]_r, \quad \text{mod } p \quad (\S\, 10),$$

$$\text{(ii.)}\quad r^p A'_p\left(\frac{l}{r}\right) \equiv \left(\frac{l}{p}\right)_{2r}, \quad \text{mod } p \quad (\S\, 23).$$

These results combined with those obtained in a previous paper have enabled us to assign the residues of $r^{2n+1}B_{2n+1}\left(\frac{l}{r}\right)$ and $r^{2n+1}A_{2n+1}\left(\frac{l}{r}\right)$ for any uneven Staudt factor for n as modulus (§§ 11, 24).

The principal particular cases of the formulæ (i.) and (ii.) are

$$(1)\quad E_{\frac{1}{2}(p-1)} \equiv 0 \text{ or } -2, \quad \text{mod } p,$$

according as p is of the form $4k+1$ or $4k+3$ (§ 14).

$$(2)\quad I_{\frac{1}{2}(p-1)} \equiv 0 \text{ or } (-1)^{\frac{1}{2}(p-1)}, \quad \text{mod } p,$$

according as p is of the form $3k+1$ or $3k+2$ (§ 15).

$$(3)\quad P_{\frac{1}{2}(p-1)} \equiv 0 \text{ or } (-1)^{\frac{1}{2}(p-1)}\, 2, \quad \text{mod } p,$$

according as p is of the forms $8k+1$ and $8k+3$, or of the forms $8k+5$ and $8k+7$ (§ 30).

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* They are $P_1 = 3$, $P_2 = 57$, $P_3 = 2763$, $P_4 = 250737$, $P_5 = 36581523$.

33. I now proceed to consider the expansions in which the quantities $a^{2n+1}B_{2n+1}\left(\frac{b}{a}\right)$ and $a^{2n+1}A'_{2n+1}\left(\frac{b}{a}\right)$ are the coefficients, in order to determine the most general expansions in which the residues of the coefficients can be assigned by the formulæ (i.) and (ii.). We know that

$$\frac{a}{2}\,\frac{\sin(2b-a)x}{\sin ax}=\frac{2b-a}{2}-a^3B_3\left(\frac{b}{a}\right)\frac{(2x)^2}{2!}+a^5B_5\left(\frac{b}{a}\right)\frac{(2x)^4}{4!}-\&\text{c.}^*$$

Putting $2b-a=c$, so that $b=\frac{1}{2}(c+a)$, we have

$$\frac{a}{2}\,\frac{\sin cx}{\sin ax}=\frac{c}{2}-a^3B_3\left(\frac{a+c}{2a}\right)\frac{(2x)^2}{2!}+a^5B_5\left(\frac{a+c}{2a}\right)\frac{(2x)^4}{4!}-\&\text{c.}$$

Similarly, we have

$$\frac{a}{2}\,\frac{\cos(2b-a)x}{\cos ax}=aA'_1\left(\frac{b}{a}\right)-a^3A'_3\left(\frac{b}{a}\right)\frac{(2x)^2}{2!}+a^5A'_5\left(\frac{b}{a}\right)\frac{(2x)^4}{4!}-\&\text{c.},^\dagger$$

giving

$$\frac{a}{2}\,\frac{\cos cx}{\cos ax}=\frac{a}{2}-a^3A'_3\left(\frac{a+c}{2a}\right)\frac{(2x)^2}{2!}+a^5A'_5\left(\frac{a+c}{2a}\right)\frac{(2x)^4}{4!}-\&\text{c.}$$

We thus find, a and c being unrestricted,

$$\frac{a}{2}\,\frac{\sin cx}{\sin ax}=\frac{c}{2}-\Delta_1\frac{(2x)^2}{2!}+\Delta_2\frac{(2x)^4}{4!}-\&\text{c.},$$

$$\frac{a}{2}\,\frac{\cos cx}{\cos ax}=\frac{a}{2}-\Theta_1\frac{(2x)^2}{2!}+\Theta_2\frac{(2x)^4}{4!}-\&\text{c.},$$

where $$\Delta_n=a^{2n+1}B_{2n+1}\left(\frac{a+c}{2a}\right),$$

$$\Theta_n=a^{2n+1}A'_{2n+1}\left(\frac{a+c}{2a}\right).$$

Since $$B_{2n+1}(1-x)=-B_{2n+1}(x),$$

and $$A'_{2n+1}(1-x)=A'_{2n+1}(x),$$

we have also $$\Delta_n=-a^{2n+1}B_{2n+1}\left(\frac{a-c}{2a}\right),$$

$$\Theta_n=a^{2n+1}A'_{2n+1}\left(\frac{a-c}{2a}\right).$$

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* *Proc. Lond. Math. Soc.*, Vol. xxxi., p. 207.

† *Ib.*, p. 206.

34. If a and c are both even integers or both uneven integers, so that $\frac{1}{2}(a+c)$ is an integer, the residues of $\Delta_{\frac{1}{2}(p-1)}$ and $\Theta_{\frac{1}{2}(p-1)}$ are given directly by (i.) and (ii.), viz.,

$$\Delta_{\frac{1}{2}(p-1)} \equiv \tfrac{1}{2}(a+c) - \left[\frac{\frac{1}{2}(a+c)}{p}\right]_a, \quad \text{mod } p,$$

$$\Theta_{\frac{1}{2}(p-1)} \equiv \left(\frac{\frac{1}{2}(a+c)}{p}\right)_{2a}, \quad \text{mod } p;$$

but, if a and c are one even and one uneven, we have to use the mulæ (i.) and (ii.) in a slightly modified form.

35. To obtain this modified form, put $r = ma$, $l = b$ in (i.) and (ii.) of § 32. We thus find

$$m^p a^p B_p \left(\frac{b}{ma}\right) \equiv b - \left[\frac{b}{p}\right]_{ma}, \quad \text{mod } p,$$

$$m^p a^p A'_p \left(\frac{b}{ma}\right) \equiv \left(\frac{b}{p}\right)_{2ma}, \quad \text{mod } p,$$

whence

$$\text{(i.)}\ a^p B_p \left(\frac{b}{ma}\right) \equiv \frac{1}{m}\left\{b - \left[\frac{b}{p}\right]_{ma}\right\}, \quad \text{mod } p,$$

$$\text{(ii.)}\ a^p A'_p \left(\frac{b}{ma}\right) \equiv \frac{1}{m}\left(\frac{b}{p}\right)_{2ma}, \quad \text{mod } p.$$

36. Applying these formulæ to the case when a and c are one even and one uneven, we have

$$\Delta_{\frac{1}{2}(p-1)} \equiv \tfrac{1}{2}\left\{a+c - \left[\frac{a+c}{p}\right]_{2a}\right\}, \quad \text{mod } p,$$

$$\Theta_{\frac{1}{2}(p-1)} \equiv \tfrac{1}{2}\left(\frac{a+c}{p}\right)_{4a}, \quad \text{mod } p.$$

These formulæ also include the case of $a+c$ even (§ 34), for, if b is divisible by any number m, $= m\beta$ say, then, by §§ 19 and 23,

$$\frac{1}{m}\left[\frac{b}{p}\right]_{ma} = \left[\frac{\beta}{p}\right]_a,$$

$$\frac{1}{m}\left(\frac{b}{p}\right)_{2ma} = \left[\frac{\beta}{p}\right]_{2a}.$$

37. In general, therefore, if p be any uneven Staudt factor for n,

$$\Delta_n \equiv \tfrac{1}{2}\left\{a+c-\left[\frac{a+c}{p}\right]\right\}_{2a}, \quad \text{mod } p,$$

whether $a+c$ be even or uneven. If $a+c$ is even, we may replace $\left[\frac{a+c}{p}\right]_{2a}$ by $2\left[\frac{\frac{1}{2}(a+c)}{p}\right]_a$.

38. Similarly, if p be any uneven Staudt factor for n,

$$\Theta_n \equiv \tfrac{1}{2}\left(\frac{a+c}{p}\right)_{4a}, \quad \text{mod } p,$$

whether $a+c$ be even or uneven. If $a+c$ is even, the congruence may be written

$$\Theta_n \equiv \left(\frac{\frac{1}{2}(a+c)}{p}\right)_{2a}, \quad \text{mod } p.$$

Thus the residue of Θ_n, mod p, can only have the values 0 and a; and it has the former or latter of these values according as $\left[\frac{a+c}{p}\right]_{4a} \leqq$ or $> 2a$.*

II. *Residues of certain Series containing* $\frac{1}{2}(p-3)$ *terms* (§§ 39–65).

39. In Vol. XXXI., p. 214, of the *Proceedings*, a general congruence theorem was given connecting the first $\frac{1}{2}(p-1)$ coefficients in a general expansion formula; and the two following formulæ were given as particular cases:—

$$\text{(i.)}\ E_0-E_1+E_2-\ldots+(-1)^{\frac{1}{2}(p-1)}E_{\frac{1}{2}(p-1)} \equiv 0, \quad \text{mod } p,$$

$$\text{(ii.)}\ I_0-I_1+I_2-\ldots+(-1)^{\frac{1}{2}(p-1)}\tfrac{3}{2}I_{\frac{1}{2}(p-1)} \equiv 0, \quad \text{mod } p,$$

the last term in the second series having the coefficient $\frac{3}{2}$. The values to be assigned to E_0 and I_0 are respectively 1 and $\frac{1}{2}$.

Since $E_{\frac{1}{2}(p-1)} \equiv 0$ or -2, mod p, according as p is of the form $4k+1$ or $4k+3$ (§ 14) and $I_{\frac{1}{2}(p-1)} \equiv 0$ or $(-1)^{\frac{1}{2}(p-1)}$, mod p, according as p is of the form $3k+1$ or $3k+2$ (§ 15), these formulæ show that

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* The formulæ for the residues of the Δ's and Θ's require modification, if $p \leqq c-a$ (*see notes to* §§ 10, 19).

$$\text{(i.)}\ E_0-E_1+E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}E_{\frac{1}{2}(p-3)}\equiv 0 \text{ or } -2, \text{ mod } p,$$

according as p is of the form $4k+1$ or $4k+3$, and

$$\text{(ii.)}\ I_0-I_1+I_2-\ldots+(-1)^{\frac{1}{2}(p-3)}I_{\frac{1}{2}(p-3)}\equiv 0 \text{ or } -\tfrac{3}{2}, \text{ mod } p,$$

according as p is of the form $3k+1$ or $3k+2$.

40. These results may be generalised by means of the formulæ (i.) and (ii.) of § 32, so that we may obtain the residues of similar expressions in which the coefficients of the expansion are multiplied by successive even powers of any number m; *e.g.*, we may assign the residue, mod p, of

$$E_0-m^2E_1+m^4E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}m^{p-3}E_{\frac{1}{2}(p-3)}.$$

41. To obtain these general results we start with the formula

$$\tfrac{1}{2}\left\{r^pB_p\left(\frac{rx+1}{r}\right)-r^pB_p\left(\frac{rx+r-1}{r}\right)\right\}$$

$$=r^pB_p\left(\frac{1}{r}\right)+(p-1)_2(rx)^2r^{p-2}B_{p-2}\left(\frac{1}{r}\right)$$

$$+(p-1)_4(rx)^4r^{p-4}B_{p-4}\left(\frac{1}{r}\right)+\&\text{c.},$$

where $(p-1)_t$ denotes the number of combinations of $p-1$ things taken t together.[*]

Since $(p-1)_t\equiv 1$, mod p, we find, by transposing to the left-hand side the first term on the right-hand side, and writing the terms on the right-hand side in the reverse order,

$$\tfrac{1}{2}\left\{r^pB_p\left(\frac{rx+1}{r}\right)-r^pB_p\left(\frac{rx+r-1}{r}\right)\right\}-r^pB_p\left(\frac{1}{r}\right)$$

$$\equiv(rx)^{p-1}rB_1\left(\frac{1}{r}\right)+(rx)^{p-3}r^3B_3\left(\frac{1}{r}\right)+\ldots+(rx)^2r^{p-2}B_{p-2}\left(\frac{1}{r}\right), \text{ mod } p$$

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[*] This formula may be derived from the first formula on p. 155 of Vol. xxix. of the *Quarterly Journal* by putting $n=p$, using the relation

$$V_{2n+1}(x)=(2n+1)B_{2n+1}(x)$$

to replace V's by B's, and dividing throughout by p.

Putting $rx = \dfrac{1}{m}$, and noticing that $m^{p-1} \equiv 1$, mod p, we obtain the formula

$$rB_1\left(\frac{1}{r}\right) + m^2r^3B_3\left(\frac{1}{r}\right) + m^4r^5B_5\left(\frac{1}{r}\right) + \ldots + m^{p-3}r^{p-2}B_{p-2}\left(\frac{1}{r}\right)$$

$$\equiv \tfrac{1}{2}\left\{r^pB_p\left(\frac{m+1}{rm}\right) - r^pB_p\left(\frac{(r-1)\,m+1}{rm}\right)\right\} - r^pB_p\left(\frac{1}{r}\right), \text{ mod } p.$$

42. Now, by § 32, the right-hand side

$$\equiv \frac{1}{2m}\left\{m+1-\left[\frac{m+1}{p}\right]_{rm} - (r-1)\,m - 1 + \left[\frac{(r-1)\,m+1}{p}\right]_{rm}\right\}$$

$$-1+\left[\frac{1}{p}\right]_r, \quad \text{mod } p,$$

$$\equiv \frac{1}{2m}\left\{\left[\frac{(r-1)\,m+1}{p}\right]_{rm} - \left[\frac{m+1}{p}\right]_{rm} - rm\right\} + \left[\frac{1}{p}\right]_r, \quad \text{mod } p;$$

and we thus obtain the formula

$$rB_1\left(\frac{1}{r}\right) + m^2r^3B_3\left(\frac{1}{r}\right) + m^4r^5B_5\left(\frac{1}{r}\right) + \ldots + m^{p-3}r^{p-2}B_{p-2}\left(\frac{1}{r}\right)$$

$$\equiv -\frac{r}{2} + \frac{1}{2m}\left\{\left[\frac{(r-1)\,m+1}{p}\right]_{rm} - \left[\frac{m+1}{p}\right]_{rm}\right\} + \left[\frac{1}{p}\right]_r, \quad \text{mod } p.$$

Thus the residue of the series depends upon r and m and upon the residue of p, mod rm.

43. Since $2^{2n+1}B_{2n+1}(\tfrac{1}{2}) = 0$, the left-hand side is zero when $r = 2$; and it is easy to see that the right-hand side also is equal to zero.

44. Since $$4^{2n+1}B_{2n+1}(\tfrac{1}{4}) = (-1)^{n+1}E_n, \quad (\S\ 12),$$

we find, by putting $r = 4$,

$$E_0 - m^2E_1 + m^4E_2 - \ldots + (-1)^{\frac{1}{2}(p-3)}E_{\frac{1}{2}(p-3)}$$

$$\equiv 2 + \frac{1}{2m}\left\{\left[\frac{m+1}{p}\right]_{4m} - \left[\frac{3m+1}{p}\right]_{4m}\right\} - \left[\frac{1}{p}\right]_4, \quad \text{mod } p,$$

which gives the residue of

$$E_0 - m^2E_1 + m^4E_2 - \ldots + (-1)^{\frac{1}{2}(p-3)}m^{p-3}E_{\frac{1}{2}(p-3)}$$

for all integral values of m.

45. For $m = 1$ the right-hand side of the congruence

$$= 2+\tfrac{1}{2}\left\{\left[\frac{2}{p}\right]_4-\left[\frac{4}{p}\right]_4\right\}-\left[\frac{1}{p}\right]_4$$

$$= 2+\tfrac{1}{2}\left\{2\left[\frac{1}{p}\right]_2-4\left[\frac{1}{p}\right]_1\right\}-\left[\frac{1}{p}\right]_4$$

$$= \left[\frac{1}{p}\right]_2-\left[\frac{1}{p}\right]_4.$$

Now $\left[\frac{1}{p}\right]_2$ always $= 1$, and $\left[\frac{1}{p}\right]_4 = 1$ or 3, according as p is of the form $4k+1$ or $4k+3$. Therefore

$$E_0-E_1+E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}E_{\frac{1}{2}(p-3)} \equiv 0 \text{ or } -2, \text{ mod } p,$$

according as p is of the form $4k+1$ or $4k+3$. This formula was given in § 39.

46. Putting $m = 2$ in § 44, we have

$$E_0-2^2E_1+2^4E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}2^{p-3}E_{\frac{1}{2}(p-3)}$$
$$\equiv 2+\tfrac{1}{4}\left\{\left[\frac{3}{p}\right]_8-\left[\frac{7}{p}\right]_8\right\}-\left[\frac{1}{p}\right]_4, \text{ mod } p.$$

The residue $\left[\frac{1}{p}\right]_4 = 1$ or 3, according as p is of the form $4k+1$ or $4k+3$. The other residues are given for the different forms of p by the following congruences :—

if $p = 8k+1$, $x \equiv 3$, mod 8, giving $x_1 = 3$,*
" " $x \equiv 7$, " " $x_1 = 7$;
if $p = 8k+3$, $3x \equiv 3$, mod 8, giving $x_1 = 1$,
" " $3x \equiv 7$, " " $x_1 = 5$;
if $p = 8k+5$, $5x \equiv 3$, mod 8, giving $x_1 = 7$,
" " $5x \equiv 7$, " " $x_1 = 3$;
if $p = 8k+7$, $7x \equiv 3$, mod 8, giving $x_1 = 5$,
" " $7x \equiv 7$, " " $x_1 = 1$.

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* x_1 will always be used to denote the least root of the congruence in question.

The residue of the series is therefore, in the four cases,

$$2+\tfrac{1}{4}(3-7)-1=0,$$
$$2+\tfrac{1}{4}(1-5)-3=-2,$$
$$2+\tfrac{1}{4}(7-3)-1=2,$$
$$2+\tfrac{1}{4}(5-1)-3=0,$$

respectively. Therefore

$$E_0-2^2E_1+2^4E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}2^{p-3}E_{\frac{1}{2}(p-3)}\equiv 0,\ -2,\ 2,\ 0, \text{ mod } p,$$

according as $p\equiv 1, 3, 5, 7$, mod 8.

47. Putting $m=3$, we have

$$E_0-3^2E_1+3^4E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}3^{p-2}E_{\frac{1}{2}(p-3)}$$
$$\equiv 2+\tfrac{1}{3}\left\{2\left[\frac{1}{p}\right]_3-\left[\frac{5}{p}\right]_6\right\}-\left[\frac{1}{p}\right]_4, \text{ mod } p.$$

The congruences giving the first two residues are

if $p=12k+1$, $x\equiv 1$, mod 3, giving $x_1=1$,
" " $x\equiv 5$, mod 6, " $x_1=5$;
if $p=12k+5$, $5x\equiv 1$, mod 3, giving $x_1=2$,
" " $5x\equiv 5$, mod 6, " $x_1=1$;
if $p=12k+7$, $7x\equiv 1$, mod 3, giving $x_1=1$,
" " $7x\equiv 5$, mod 6, " $x_1=5$;
if $p=12k+11$, $11x\equiv 1$, mod 3, giving $x_1=2$,
" " $11x\equiv 5$, mod 6, " $x_1=1$.

Thus in the four cases the residue of the series $=0, 2, -2, 0$, respectively; and therefore

$$E_0-3^2E_1+3^4E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}3^{p-3}E_{\frac{1}{2}(p-3)}\equiv 0,\ 2,\ -2,\ 0, \text{ mod } p,$$

according as $p\equiv 1, 5, 7, 11$, mod 12.

48. Similarly, putting $m=4$,

$$E_0-4^2E_1+4^4E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}4^{p-2}E_{\frac{1}{2}(p-3)}$$
$$\equiv 2+\tfrac{1}{8}\left\{\left[\frac{5}{p}\right]_{16}-\left[\frac{13}{p}\right]_{16}\right\}-\left[\frac{1}{p}\right]_4, \text{ mod } p.$$

By considering as in the two preceding sections the values of the residues according to the different forms of p, we find that

$$E_0-4^2E_1+4^4E_2-\dots+(-1)^{\frac{1}{2}(p-3)}4^{p-3}E_{\frac{1}{2}(p-3)}$$
$$\equiv 0,\ -2,\ 0,\ -2,\ 2,\ 0,\ 2,\ 0,\ \bmod p,$$

according as $p \equiv 1, 3, 5, 7, 9, 11, 13, 15, \bmod 16$.

49. Putting now $r=3$ in the general congruence of § 42, and using the formula

$$3^{2n+1}B_{2n+1}\left(\tfrac{1}{3}\right)=(-1)^n I_n \ (\S\ 12),$$

we find

$$I_0-m^2I_1+m^4I_2-\dots+(-1)^{\frac{1}{2}(p-3)}m^{p-2}I_{\frac{1}{2}(p-3)}$$
$$\equiv \tfrac{3}{2}+\frac{1}{2m}\left\{\left[\frac{m+1}{p}\right]_{3m}-\left[\frac{2m+1}{p}\right]_{3m}\right\}-\left[\frac{1}{p}\right]_3,\ \bmod p.$$

50. By putting $m=1, 2, 3, 4$, and determining the separate residues exactly as in the case of $r=4$, we obtain the following results:—

(i.) $I_0-I_1+I_2-\dots+(-1)^{\frac{1}{2}(p-3)}I_{\frac{1}{2}(p-3)} \equiv 0$ or $-\tfrac{3}{2}$, $\bmod p$,

according as p is of the form $3k+1$ or $3k+2$. This formula has been already given in § 39.

(ii.) $I_0-2^2I_1+2^4I_2-\dots+(-1)^{\frac{1}{2}(p-3)}2^{p-3}I_{\frac{1}{2}(p-3)} \equiv 0,\ \bmod p$,

for all values of p.

(iii.) $I_0-3^2I_1+3^4I_2-\dots+(-1)^{\frac{1}{2}(p-3)}3^{p-3}I_{\frac{1}{2}(p-3)}$

$$\equiv 0,\ -\tfrac{3}{2},\ 0,\ 0,\ \tfrac{3}{2},\ 0,\ \bmod p,$$

according as $p \equiv 1, 2, 4, 5, 7, 8, \bmod 9$.

(iv.) $I_0-4^2I_1+4^4I_2-\dots+(-1)^{\frac{1}{2}(p-3)}4^{p-3}I_{\frac{1}{2}(p-3)}$

$$\equiv 0,\ -\tfrac{3}{2},\ \tfrac{3}{2},\ 0,\ \bmod p,$$

according as $p \equiv 1, 5, 7, 11, \bmod 12$.

51. It may be remarked that in general, if p is of the form $krm+1$,

$$rB_1\left(\frac{1}{r}\right)+m^2r^3B_3\left(\frac{1}{r}\right)+m^4r^5B_5\left(\frac{1}{r}\right)+\dots$$
$$\dots+m^{p-3}r^{p-2}B_{p-2}\left(\frac{1}{r}\right)\equiv 0,\ \bmod p;$$

for in this case the value of the right-hand side of the formula in § 42

$$= -\frac{r}{2} + \frac{1}{2m}\{(r-1)m+1-(m+1)\}+1 = 0.$$

52. The A'-formula corresponding to the B-formula of § 41 is

$$\tfrac{1}{2}\left\{r^p A_p'\left(\frac{rx+1}{r}\right) + r^p A_p'\left(\frac{rx+r-1}{r}\right)\right\} - r^p A_p'\left(\frac{1}{r}\right)$$

$$\equiv (rx)^{p-1} rA_1'\left(\frac{1}{r}\right) + (rx)^{p-3} r^3 A_3'\left(\frac{1}{r}\right) + \ldots + (rx)^2 r^{p-2} A_{p-2}'\left(\frac{1}{r}\right), \text{ mod } p,$$

whence, putting $rx = \dfrac{1}{m}$,

$$rA_1'\left(\frac{1}{r}\right) + m^2r^3A_3'\left(\frac{1}{r}\right) + m^4r^5A_5'\left(\frac{1}{r}\right) + \ldots + m^{p-3}r^{p-2}A_{p-2}'\left(\frac{1}{r}\right)$$

$$\equiv \tfrac{1}{2}\left\{r^p A_p'\left(\frac{m+1}{rm}\right) + r^p A_p'\left(\frac{(r-1)m+1}{r}\right)\right\} - r^p A_p'\left(\frac{1}{r}\right), \text{ mod } p.$$

53. By § 35 the right-hand side

$$\equiv \frac{1}{2m}\left\{\left(\frac{m+1}{p}\right)_{2rm} + \left(\frac{(r-1)m+1}{p}\right)_{2rm}\right\} - \left(\frac{1}{p}\right)_{2r}, \text{ mod } p;$$

and therefore we find that

$$rA_1'\left(\frac{1}{r}\right) + m^2r^3A_3'\left(\frac{1}{r}\right) + m^4r^5A_5'\left(\frac{1}{r}\right) + \ldots + m^{p-3}r^{p-2}A_{p-2}'\left(\frac{1}{r}\right)$$

$$\equiv \frac{A+B}{2m} - C, \text{ mod } p,$$

where $A = 0$ or rm according as the least root of $px \equiv m+1$, mod $2rm$, is $\leqq$ or $> rm$; $B = 0$ or rm, according as the least root of $px \equiv (r-1)m+1$, mod $2rm$, is $\leqq$ or $> rm$; and $c = 0$ or r, according as the least root of $px \equiv 1$, mod $2r$, is $\leqq$ or $> r$.

54. Since $2^{2n+1}A_{2n+1}'(\tfrac{1}{2}) = (-1)^n E_n$ (§ 25),
the formula becomes, by putting $r = 2$,

$$E_0 - m^2E_1 + m^4E_2 - \ldots + (-1)^{\frac{1}{2}(p-3)} m^{p-3} E_{\frac{1}{2}(p-3)} \equiv \frac{A}{m} - C, \text{ mod } p,$$

where $\quad A = 0$ or $2m$ according as $\left[\frac{m+1}{p}\right]_{4m} <$ or $> 2m$,

$\qquad C = 0$ or $2 \qquad$ „ $\quad$ „ $\quad \left[\frac{1}{p}\right]_{4} <$ or > 2.

Thus $C = 0$ or 2, according as p is of the form $4k+1$ or $4k+3$.

55. When $m = 1$, $A = \left(\frac{2}{p}\right)_4 = 0$, and therefore the series $\equiv 0$ or -2, mod p, according as p is of the form $4k+1$ or $4k+3$.

When $m = 2$, $A = \left(\frac{3}{p}\right)_8$, which $= 0, 0, 4, 4$, according as $p \equiv 1, 3, 5, 7$, mod 8. Therefore the series $\equiv 0, -2, 2, 0$, mod p, according as $p \equiv 1, 3, 5, 7$, mod 8.

When $m = 3$, $A = \left(\frac{4}{p}\right)_{12} = 0$ or 6, according as p is of the form $3k+1$ or $3k+2$. Therefore the series $\equiv 0, 2, -2, 0$, mod p, according as $p \equiv 1, 5, 7, 11$, mod 12.

When $m = 4$, $A = \left(\frac{5}{p}\right)_{16}$, which $= 0$ if $p \equiv 1, 3, 5, 7$, mod 16, and $= 8$ if $p = 9, 11, 13, 15$, mod 16. Therefore the series $\equiv 0, -2, 0, -2, 2, 0, 2, 0$, mod p, according as $p \equiv 1, 3, 5, 7, 9, 11, 13, 15$, mod 16.

These results agree with those already obtained in §§ 45–48. It will be seen that the second formula (§ 53) is simpler than the first* (§ 44).

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* Comparing the two formulæ of §§ 44 and 53, we find that they lead to the equation

$$\left[\frac{m+1}{p}\right]_{4m} + \left[\frac{3m+1}{p}\right]_{4m} = \left[\frac{2m+2}{p}\right]_{4m} + 2m.$$

This relation may be proved independently; for, if $p = 4km+h$, h being $< 4m$ and prime to it, and if a_1, b_1, c_1 be the least roots of the respective congruences

$$hx \equiv m+1, \quad hx \equiv 3m+1, \quad hx \equiv 2m+2, \quad \text{mod } 4m,$$

then $$h(a_1+b_1-c_1) = 2m+\lambda . 4m,$$

whence $$a_1+b_1-c_1 = \frac{2\lambda+1}{h} 2m = \nu . 2m,$$

ν being an uneven integer. Now it is obvious that ν cannot be > 1, for a_1, b_1, c_1 are all $\overset{=}{<} 4m$, so that $a_1+b_1-c_1$ cannot exceed $8m$, and therefore we must have $\nu = 1$.

56. Since $$4^{2n+1}A'_{2n+1}(\tfrac{1}{4}) = (-1)^n 2P_n \text{ (§ 25)},$$
we find, by putting $r=4$,

$$P_0 - m^2P_1 + m^4P_2 - \dots + (-1)^{\frac{1}{2}(p-3)} m^{p-3} P_{\frac{1}{2}(p-3)}\text{[*]}$$
$$= \frac{1}{4m}\left\{\left(\frac{m+1}{p}\right)_{8m} + \left(\frac{3m+1}{p}\right)_{8m}\right\} - \tfrac{1}{2}\left(\frac{1}{p}\right)_8, \text{ mod } p,$$
$$= \frac{A+B}{4m} - C, \text{ mod } p,$$

where $A = 0$ or $4m$, according as $\left[\dfrac{m+1}{p}\right]_{8m} <$ or $> 4m$,

$B = 0$ or $4m$, „ „ $\left[\dfrac{3m+1}{p}\right]_{8m} <$ or $> 4m$,

$C = 0$ or 4, „ „ $\left[\dfrac{1}{p}\right]_8 <$ or > 4.

As particular cases we find

(i.) $P_0 - P_1 + P_2 - \dots + (-1)^{\frac{1}{2}(p-3)} P_{\frac{1}{2}(p-3)} \equiv 0, 1, -2, -1$, mod p,
according as $p \equiv 1, 3, 5, 7$, mod 8.

(ii.) $P_0 - 2^2P_1 + 2^4P_2 - \dots + (-1)^{\frac{1}{2}(p-3)} 2^{p-3} P_{\frac{1}{2}(p-3)}$
$\equiv 0, 1, -1, -2, 2, 1, -1, 0$, mod p,
according as $p \equiv 1, 3, 5, 7, 9, 11, 13, 15$, mod 16.

(iii.) $P_0 - 3^2P_1 + 3^4P_2 - \dots + (-1)^{\frac{1}{2}(p-3)} 3^{p-3} P_{\frac{1}{2}(p-3)}$
$\equiv 0, -1, -1, 2, -2, 1, 1, 0$, mod p,
according as $p \equiv 1, 5, 7, 11, 13, 17, 19, 23$, mod 24.

57. For $r = 3$ the formula of § 55 gives

$$H_0 - m^2H_1 + m^4H_2 - \dots + (-1)^{\frac{1}{2}(p-3)} m^{p-3} H_{\frac{1}{2}(p-3)}\text{[†]}$$
$$\equiv \frac{1}{2m}\left\{\left(\frac{m+1}{p}\right)_{6m} + \left(\frac{2m+1}{p}\right)_{6m}\right\} - \left(\frac{1}{p}\right)_6, \text{ mod } p,$$

from which we obtain the particular cases:

(i.) $H_0 - H_1 + H_2 - \dots + (-1)^{\frac{1}{2}(p-3)} H_{\frac{1}{2}(p-3)} \equiv 0$ or $-\frac{3}{2}$, mod p,
according as p is of the form $3k+1$ or $3k+2$.

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[*] The value of P_0 is 1 (§ 25).

[†] The value of H_0 is $\frac{3}{2}$ (§ 25). The values of H_n up to $n = 13$ were given in *Proc. Lond. Math. Soc.*, Vol. XXXI., p. 229.

(ii.) $H_0 - 2^2H_1 + 2^4H_2 - \ldots + (-1)^{\frac{1}{2}(p-3)}2^{p-3}H_{\frac{1}{2}(p-3)}$

$$\equiv 0,\ -3,\ 3,\ 0,\ \text{mod}\ p,$$

according as $p \equiv 1, 5, 7, 11$, mod 12.

(iii.) $H_0 - 3^2H_1 + 3^4H_2 - \ldots + (-1)^{\frac{1}{2}(p-3)}3^{p-3}H_{\frac{1}{2}(p-3)}$

$$\equiv 0,\ -3,\ \tfrac{3}{2},\ -\tfrac{3}{2},\ 3,\ 0,\ \text{mod}\ p,$$

according as $p \equiv 1, 5, 7, 11, 13, 17$, mod 18.

58. If p is of the form $2krm+1$, the right-hand member of the general formula is zero, for in this case each of the quantities A, E, C (§ 53) is zero, so that we have

$$rA_1'\left(\frac{1}{r}\right) + m^2r^3A_3'\left(\frac{1}{r}\right) + m^4r^5A_5'\left(\frac{1}{r}\right) + \ldots + m^{p-3}A'_{p-2}\left(\frac{1}{r}\right) \equiv 0,\ \text{mod}\ p.$$

The corresponding B-formula was given in § 51.

59. I have verified numerically all the formulæ involving E's, Γs, P's, and H's in §§ 45–50 and §§ 55–57 for the values 5, 7, 11, 13 of p.

60. If in §§ 41 and 52 we put $rx = m$ $\left(\text{instead of } \frac{1}{m}\right)$, we obtain similar series, but in which the coefficients are descending powers of m^2, viz., m^{p-1}, m^{p-3}, m^{p-5}, …, m^2. Since $m^{p-1} \equiv 1$, mod p, we may also write these coefficients $1, \frac{1}{m^2}, \frac{1}{m^4}, \ldots, \frac{1}{m^{p-3}}$.

We thus find

$$rB_1\left(\frac{1}{r}\right) + \frac{1}{m^2}r^3B_3\left(\frac{1}{r}\right) + \frac{1}{m^4}r^5B_5\left(\frac{1}{r}\right) + \ldots + \frac{1}{m^{p-3}}r^{p-2}B_{p-2}\left(\frac{1}{r}\right)$$

$$\equiv \tfrac{1}{2}\left\{r^pB_p\left(\frac{m+1}{r}\right) - r^pB_p\left(\frac{m+r-1}{r}\right)\right\} - r^pB_p\left(\frac{1}{r}\right),\ \text{mod}\ p,$$

$$rA_1'\left(\frac{1}{r}\right) + \frac{1}{m^2}r^3A_3'\left(\frac{1}{r}\right) + \frac{1}{m^4}r^5A_5'\left(\frac{1}{r}\right) + \ldots + \frac{1}{m^{p-3}}r^{p-2}A'_{p-2}\left(\frac{1}{r}\right)$$

$$\equiv \tfrac{1}{2}\left\{r^pA_p'\left(\frac{m+1}{r}\right) + r^pA_p'\left(\frac{m+r-1}{r}\right)\right\} - r^pA_p'\left(\frac{1}{r}\right),\ \text{mod}\ p;$$

whence, by § 32,

$$\text{(i.)}\ rB_1\left(\frac{1}{r}\right)+\frac{1}{m^2}r^3B_3\left(\frac{1}{r}\right)+\frac{1}{m^4}r^5B_5\left(\frac{1}{r}\right)+\ldots+\frac{1}{m^{p-3}}r^{p-2}B_{p-2}\left(\frac{1}{r}\right)$$

$$\equiv -\frac{r}{2}+\tfrac{1}{2}\left\{\left[\frac{m+r-1}{p}\right]_r-\left[\frac{m+1}{p}\right]_r\right\}+\left[\frac{1}{p}\right]_r, \text{ mod } p.$$

$$\text{(ii.)}\ rA'_1\left(\frac{1}{r}\right)+\frac{1}{m^2}r^3A'_3\left(\frac{1}{r}\right)+\frac{1}{m^4}r^5A'_5\left(\frac{1}{r}\right)+\ldots+\frac{1}{m^{p-3}}r^{p-2}A'_{p-2}\left(\frac{1}{r}\right)$$

$$\equiv \tfrac{1}{2}\left\{\left(\frac{m+1}{p}\right)_{2r}+\left(\frac{m+r-1}{p}\right)_{2r}\right\}-\left(\frac{1}{p}\right)_{2r}, \text{ mod } p.$$

61. It will be seen that in the formulæ (i.) and (ii.) the classification of p is much simpler than in the corresponding formulæ of §§ 42 and 53, viz., the different cases depend upon the residue of p, mod r or mod $2r$, instead of upon the residue of p, mod mr or mod $2mr$. Thus the number of cases is only $\phi(r)$ or $\phi(2r)$ instead of $\phi(mr)$ or $\phi(2mr)$, where $\phi(n)$ denotes the number of numbers less than n and prime to it. Not only therefore is the number of cases smaller, but the cases themselves are independent of m; so that for any given values of p and m the residue of the series, mod p, depends only upon the residues of p and m, mod r or mod $2r$.

62. Putting $r = 4$ in (i.), we have

$$E_0-\frac{1}{m^2}E_1+\frac{1}{m^4}E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}\frac{1}{m^{p-3}}E_{\frac{1}{2}(p-3)}$$

$$\equiv 2+\tfrac{1}{2}\left\{\left[\frac{m+1}{p}\right]_4-\left[\frac{m+3}{p}\right]_4\right\}-\left[\frac{1}{p}\right]_4, \text{ mod } p.$$

Denoting the right-hand side by R, we find, by putting successively $m = 1, 2, 3, 4$, that, if $p = 4k+1$, then

$$R = 0,\ 2,\ 2,\ 0, \text{ according as } m \equiv 1,\ 2,\ 3,\ 4, \text{ mod } 4,$$

and that, if $p = 4k+3$, then

$$R = -2,\ -2,\ 0,\ 0, \text{ according as } m \equiv 1,\ 2,\ 3,\ 4, \text{ mod } 4.$$

Thus the residue of

$$E_0-\frac{1}{m^2}E_1+\frac{1}{m^4}E_2-\ldots+(-1)^{\frac{1}{2}(p-3)}\frac{1}{m^{p-3}}E_{\frac{1}{2}(p-3)}$$

is given by the following table:—

	$m \equiv 1$	$m \equiv 2$	$m \equiv 3$	$m \equiv 4$
$p \equiv 1$	0	2	2	0
$p \equiv 1$	-2	-2	0	0

in which the headings of the columns are the residues of m, mod 4, and the arguments of the lines are the residues of p, mod 4.

Thus, for example, since $7 \equiv 3$, mod 4, the table shows that

$$E_0 - \frac{1}{7^2} E_1 + \frac{1}{7^4} E_2 - \ldots + (-1)^{\frac{1}{2}(p-3)} \frac{1}{7^{p-3}} E_{\frac{1}{2}(p-3)} \equiv 2 \text{ or } 0, \text{ mod } p,$$

according as p is of the form $4k+1$ or $4k+3$.

63. It is to be noticed that p must not be equal to m, and that the formula and the results given by the table require modification if $p<m$. For in § 10 it was necessary that l should be $< p+r$, which in this case gives $m+3 < p+4$, that is, $p > m-1$.

64. Putting $r = 2$, the formula (ii.) of § 60 gives

$$E_0 - \frac{1}{m^2} E_1 + \frac{1}{m^4} E_2 - \ldots + (-1)^{\frac{1}{2}(p-3)} \frac{1}{m^{p-3}} E_{\frac{1}{2}(p-3)}$$

$$\equiv \left(\frac{m+1}{p}\right)_4 - \left(\frac{1}{p}\right)_4, \quad \text{mod } p,$$

from which we may derive, and in a simpler manner, the results given in § 62.*

I have verified these results numerically for the values 1, 2, 3, 4, 5, 6, 7 of m, and the values 5, 7, 11, 13 of p.

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* Comparing the two formulæ, we obtain the relation

$$\left[\frac{m+1}{p}\right]_4 + \left[\frac{m+3}{p}\right]_4 = \left[\frac{2m+2}{p}\right]_4 + 2,$$

which may be proved independently by the same reasoning as that employed to prove the similar theorem in the note to § 55.

65. Putting $r = 3$ in (i.) of § 60, we have

$$I_0 - \frac{1}{m^2} I_1 + \frac{1}{m^4} I_2 - \ldots + (-1)^{\frac{1}{2}(p-3)} \frac{1}{m^{p-3}} I_{\frac{1}{2}(p-3)}$$

$$\equiv \tfrac{3}{2} + \tfrac{1}{2} \left\{ \left[\frac{m+1}{p}\right]_3 - \left[\frac{m+2}{p}\right]_3 \right\} - \left[\frac{1}{p}\right]_3, \text{ mod } p.$$

From this formula we obtain the following table giving the residue of the series according to the residues of m and p, mod 3:—

	$m \equiv 1$	$m \equiv 2$	$m \equiv 3$
$p \equiv 1$	0	$\frac{3}{2}$	0
$p \equiv 2$	$-\frac{3}{2}$	0	0

Thus, for example, for $m = 5$, which $\equiv 2$, mod 3, we have

$$I_0 - \frac{1}{5^2} I_1 + \frac{1}{5^4} I_2 - \ldots + (-1)^{\frac{1}{2}(p-3)} \frac{1}{m^{p-3}} I_{\frac{1}{2}(p-3)} \equiv \tfrac{3}{2} \text{ or } 0, \text{ mod } p,$$

according as p is of the form $3k+1$ or $3k+2$.

I have verified the table for $m = 1, 2, 3, 4, 5$ and $p = 5, 7, 11, 13$. The results require modification if $p < m$.

66. In this paper I have restricted myself to the consideration of the Bernoullian functions $B_n(x)$ and $A'_n(x)$, when the suffix n is uneven. I have obtained also the residues of these functions in the case when n is even, but the results are more complicated, and I reserve their consideration for a separate paper.

Note on the Quinquisectional Equation. *By* L. J. ROGERS.

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It has been shown by Gauss and Lebesgue that the tri- and quadrisectional equations for a prime p can be made to depend upon the resolution of p into certain quadratic forms, the former depending on an identity $4p = M^2+27N^2$, and the latter on an identity $p = A^2+B^2$.

It is proposed here to establish a similar method for the derivation of the quinquisectional equation for primes of the form $5\lambda+1$.

In Vol. XVIII., p. 215, Prof. Lloyd Tanner adopts the following notation:—

$$F\omega \equiv X_0+\omega X_1+\omega^2 X_2+\omega^3 X_3+\omega^4 X_4,$$

where the X's are the roots of the quinquisectional equation and ω is a fifth root of unity. Thus $F\omega = (\omega^\lambda, r)$ of Bachmann's *Kreistheilungslehre.*

Moreover, $q\omega$ denotes $F\omega^2/(F\omega)^2$, whence, by the general theorem of Jacobi's,

$$q\omega \,.\, q\omega^4 = p.$$

Thus, putting $\quad q\omega = q_0+q_1\omega+q_2\omega^2+q_3\omega^3+q_4\omega^4,$

we have (see Vol. XVIII., p. 217)

$$\begin{aligned} q_0+q_1+q_2+q_3+q_4 &= -1, \\ q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0 &= -\lambda, \\ q_0q_2+q_1q_3+q_2q_4+q_3q_0+q_4q_1 &= -\lambda, \\ q_0^2+q_1^2+q_2^2+q_3^2+q_4^2 &= 4\lambda+1. \end{aligned}$$

These give

$$\left.\begin{aligned} 16p = (4q_0-q_1-q_2-q_3-q_4)^2+5\,(q_1-q_2-q_3+q_4)^2 \\ +10\,(q_1-q_4)^2+10\,(q_2-q_3)^2 \\ \text{and } (4q_0-q_1-q_2-q_3-q_4)(q_1-q_2-q_3+q_4) \\ = -(q_1-q_4)^2-4\,(q_1-q_4)(q_2-q_3)+(q_2-q_3)^2 \end{aligned}\right\}, \quad (1)$$

$$\left.\begin{aligned} &\text{i.e.,} & 16p &= a^2+5b^2+10\,(c^2+d^2) \\ &\text{where} & ab &= -c^2-4cd+d^2 \end{aligned}\right\}, \quad (2)$$

and

$$\begin{aligned} a &= 4q_0-q_1-q_2-q_3-q_4 = 5q_0+1, \\ b &= q_1-q_2-q_3+q_4, \\ c &= q_1-q_4, \\ d &= q_2-q_3. \end{aligned}$$

Equations (2), however, are not a unique representation of p in this form, since the cyclic symmetry in the q's shows that similar expressions may be obtained by rotating the suffixes, so that a might $= 5q_1+1$, $b = q_2-q_3-q_4+q_0$, &c. Moreover, if we were to treat the trisectional complexes in a similar way, we should only obtain the representation of $4p$ in the form m^2+3n^2, which is not definite enough to establish the trisectional equation.

Passing on to p. 221 in Vol. XVIII., we find a modification of the above results, depending on the fact that

$$X_0^2 = \alpha_0 X_0+\alpha_1 X_1+\alpha_2 X_2+\alpha_3 X_3+\alpha_4 X_4,$$

where the α's are all integers.

The connexion between the α's and the q's is shown on p. 222, viz.,

$$\begin{aligned} 5\alpha_0 &= -2 \quad +3q_0-4\lambda, \\ 5\alpha_1 &= \quad 2q_1+\ q_3-4\lambda, \\ 5\alpha_2 &= \quad 2q_2+\ q_1-4\lambda, \\ 5\alpha_3 &= \quad 2q_3+\ q_4-4\lambda, \\ 5\alpha_4 &= \quad 2q_4+\ q_2-4\lambda, \end{aligned}$$

which gives $q_1-q_2-q_3+q_4 = 5\,(\alpha_1-\alpha_2-\alpha_3+\alpha_4) = 5l$, say,

$$2\,(q_1-q_4)-(q_2-q_3) = 5\,(\alpha_1-\alpha_4) \qquad = 5m, \text{ say,}$$

$$(q_1-q_4)+2\,(q_2-q_3) = 5\,(\alpha_2-\alpha_3) \qquad = 5n, \text{ say.}$$

Hence equation (2) becomes

$$\left.\begin{aligned} 16p &= a^2+125l^2+50\,(m^2+n^2) \\ al &= m^2-4mn-n^2 \end{aligned}\right\}. \tag{3}$$

We see then that every prime of the form $5\lambda+1$ can be expressed as the sixteenth of the sum of such multiples of four squares, which are connected by a single further relation.

It remains only to show that such a relation is unique, and that, if it has once been obtained, by trial or otherwise, then the quinquisectional equation can be derived from it without ambiguity. When

p is not large, (3) can be found by inspection, but, in most cases, this paper does not profess to offer an essentially simpler method than those established by Prof. Lloyd Tanner.

To show that the representation is unique, we may observe that, if $\omega = e^{\frac{2}{5}\pi i}$,

$$\begin{aligned} 16p = {} & \{a+5\sqrt{5}l+2\sqrt{5}(\omega^2-\omega^3)m+2\sqrt{5}(\omega-\omega^4)n\} \\ & \times\{a+5\sqrt{5}l-2\sqrt{5}(\omega^2-\omega^3)m-2\sqrt{5}(\omega-\omega^4)n\} \\ = {} & a^2+125l^2+50(m^2+n^2)+10\sqrt{5}(al-m^2+4mn-n^2). \end{aligned}$$

Moreover, if α, β, γ, δ be any integers, we have identically

$$\begin{aligned} & \{a+5\sqrt{5}l+2\sqrt{5}(\omega^2-\omega^3)m+2\sqrt{5}(\omega-\omega^4)n\} \\ & \times\{\alpha+5\sqrt{5}\lambda-2\sqrt{5}(\omega^2-\omega^3)\mu-2\sqrt{5}(\omega-\omega^4)\nu\} \\ = {} & (a\alpha+125l\lambda+50m\mu+50n\nu) \\ & +5\sqrt{5}(a\lambda+\alpha l-2m\mu+4m\nu+4\mu\nu+2n\nu) \\ & +2\sqrt{5}(\omega^2-\omega^3)\{-a\mu+\alpha m+5l(\mu-2\nu)-5\lambda(m-2n)\} \\ & +2\sqrt{5}(\omega-\omega^4)\{-a\nu+\alpha n--5l(2\mu+\nu)+5\lambda(2m+n)\} \\ = {} & A+5\sqrt{5}\,.\,L+2\sqrt{5}(\omega^2-\omega^3)M+2\sqrt{5}(\omega-\omega^4)N, \text{ say.} \qquad (4) \end{aligned}$$

If, then, also, $$16p = \alpha^2+125\lambda^2+50(\mu^2+\nu^2),$$

where $$\alpha\lambda = \mu^2-4\mu\nu-\nu^2,$$

we see that, by interchanging italic and Greek letters in the left-hand side of (4) and multiplying the four factors together, we have

$$\begin{aligned} 256p^2 &= \{a^2+125l^2+50(m^2+n^2)\}\{\alpha^2+125\lambda^2+50(\mu^2+\nu^2)\} \\ &= A^2+125L^2+50(M^2+N^2), \end{aligned}$$

where $$AL = M^2-LMN-N^2.$$

Again, if we change ω into ω^2, we get

$$\begin{aligned} 16p = {} & \{a-5\sqrt{5}l-2\sqrt{5}(\omega-\omega^4)m+2\sqrt{5}(\omega^2-\omega^3)n\} \\ & \times\{a-5\sqrt{5}l+2\sqrt{5}(\omega-\omega^4)m+2\sqrt{5}(\omega^2-\omega^3)n\}. \end{aligned}$$

Now, if instead of $\omega = e^{\frac{2}{5}\pi i}$, we suppose that

$$\omega^4+\omega^3+\omega^2+\omega+1 \equiv 0, \quad \text{mod } p,$$

and $$\sqrt{5} \equiv \omega+\omega^4-\omega^2-\omega^3,$$

it is clear that one of the factors in each of the products representing $16p$ must $\equiv 0$.

Without loss of generality, we may assume

$$a+5\sqrt{5}\,l \equiv 2\sqrt{5}\,(\omega^2-\omega^3)\,m+2\sqrt{5}\,(\omega-\omega^4)\,n,$$

$$a-5\sqrt{5}\,l \equiv 2\sqrt{5}\,(\omega-\omega^4)\,m-2\sqrt{5}\,(\omega^2-\omega^3)\,n,$$

and, similarly,

$$\alpha+5\sqrt{5}\,\lambda \equiv 2\sqrt{5}\,(\omega^2-\omega^3)\,\mu+2\sqrt{5}\,(\omega-\omega^4)\,\nu,$$

$$\alpha-5\sqrt{5}\,\lambda \equiv 2\sqrt{5}\,(\omega-\omega^4)\,\mu-2\sqrt{5}\,(\omega^2-\omega^3)\,\nu.$$

From these four congruences, we see that

$$(a+5\sqrt{5}\,l)(\alpha+5\sqrt{5}\,\lambda)+(a-5\sqrt{5}\,l)(\alpha-5\sqrt{5}\,\lambda)$$

$$\equiv 5\left\{4\,(\omega^2-\omega^3)^3+4\,(\omega-\omega^4)^2\right\}(m\mu+n\nu),$$

i.e., $$a\alpha+125l\lambda+50\,(m\mu+n\nu) \equiv 0,$$

while $$(a+5\sqrt{5}\,l)(\alpha+5\sqrt{5}\,\lambda)-(a-5\sqrt{5}\,l)(\alpha-5\sqrt{5}\,l)$$

$$\equiv 40\,(\omega^2-\omega^3)(\omega-\omega^4)(m\nu+\mu n)$$

$$+20\left\{(\omega^2-\omega^3)^2-(\omega-\omega^4)^2\right\}(m\mu-n\nu)$$

$$\equiv -40\sqrt{5}\,(m\nu+\mu n)+20\sqrt{5}\,(m\mu-n\nu),$$

i.e., $$a\lambda+\alpha l+4\,(m\nu+\mu n)+2m\mu-2n\nu \equiv 0.$$

Thus $$A \equiv 0 \quad \text{and} \quad L \equiv 0,$$

and, since, in consequence,

$$M^2+N^2 \equiv 0 \quad \text{and} \quad M^2-4MN-N^2 \equiv 0,$$

we see that $$M \equiv 0, \quad N \equiv 0.$$

Let $$A = Up, \quad L = Xp, \quad M = Yp, \quad N = Zp,$$

so that $$256p^2 = p^2\,(U^2+125X^2+50Y^2+50Z^2),$$

$$256 = U^2+125X^2+50Y^2+50Z^2,$$

where $$UX = Y^2-4YZ-Z^2.$$

The only possible solution for these equations is

$$U = 16, \quad X = Y = Z = 0,$$

so that $$a\alpha+125l\lambda+50\,(m\mu+n\nu) = 16p, \tag{5}$$

while $$L = M = N = 0.$$

These last three equations give the ratios of α, λ, μ, ν uniquely in terms of a, l, m, n, and, by inspection, we see

$$\alpha = ka, \quad \lambda = kl, \quad \mu = km, \quad \nu = kn.$$

But this reduces (5) to

$$k(a^2+125l^2+50m^2+50n^2) = 16p;$$

therefore $$k = 1.$$

Hence we see that the representation of $16p$ is unique.

2. We may now obtain the quinquisectional equation in terms of a, l, m, n. On p. 227, Vol. XVIII., it is given in the form

$$Y^5-10p\,.\,Y^3-5p\Sigma q\omega\, Y^2+5p\,(p-\Sigma q\omega q\omega^2)\,Y-p\Sigma q\omega q\omega q\omega^3 = 0, \quad (1)$$

where Y has for roots $5X_0+1$, $5X_1+1$, &c.

Since $$q\omega\, q\omega^4 = p,$$

we may put $$q\omega = \sqrt{p}\, e^{\theta_1 i}, \quad q\omega_4 = \sqrt{p} - e^{-\theta_1 i},$$

and, similarly, $$q\omega^2 = \sqrt{p}\, e^{\theta_2 i}, \quad q\omega^3 = \sqrt{p}\, e^{-\theta_2 i},$$

where we may suppose $\sqrt{p}$ to be the positive square root. Then

$$\left.\begin{aligned}\Sigma q\omega &= 2\sqrt{p}\,(\cos\theta_1+\cos\theta_2)\\ \Sigma q\omega q\omega^2 &= 4p\cos\theta_1\cos\theta_2\\ \Sigma q\omega q\omega q\omega^2 &= 2p\sqrt{p}\,\{\cos(2\theta_1+\theta_2)+\cos(\theta_1-2\theta_2)\}\\ &= 2p\sqrt{p}\,\{(2\cos\theta_1\cos\theta_2-1)(\cos\theta_1+\cos\theta_2)\\ &\qquad -2\sin\theta_1\sin\theta_2\,(\cos\theta_1-\cos\theta_2)\}\end{aligned}\right\}. \quad (2)$$

Moreover, if $$F\omega = \sqrt{p}\, e^{\phi_1 i}, \quad F\omega^4 = \sqrt{p}\, e^{-\phi_1 i},$$

$$F\omega^2 = \sqrt{p}\, e^{\phi_2 i}, \quad F\omega^3 = \sqrt{p}\, e^{-\phi_2 i},$$

then equations (8) and (9) on p. 217 are equivalent to

$$2\phi_1 = \theta_1+\phi_2,$$

$$2\phi_2 = \theta_2-\phi_1,$$

whence $$\phi_1 = \tfrac{1}{5}(2\theta_1+\theta_2),$$

$$\phi_2 = \tfrac{1}{5}(2\theta_2-\theta_1),$$

while $$Y_r = \omega^{-r}F\omega+\omega^{-2r}F\omega^2+\omega^{-3r}F\omega^3+\omega^{-4r}F\omega^4$$

$$= 2\sqrt{p}\left\{\cos\left(\phi_1-\frac{2\pi}{5}r\right)+\cos\left(\phi_2-\frac{4\pi}{5}r\right)\right\}, \quad (3)$$

where $r = 0, 1, 2, 3, 4$.

This shows that any equation of form (1), whether quinquisectional or not, whose coefficients depend on (2), can be solved trigonometrically by the quinquisection of an angle $2\theta_1+\theta_2$ or $2\theta_2-\theta_1$ depending on the coefficients.

Now $\sqrt{p}\,e^{\theta_1 i} = q\omega$ and $\sqrt{p}\,e^{\theta_2 i} = q\omega^2$,

whence
$$\sqrt{p}\cos\theta_1 = q_0+(q_1+q_4)\frac{\sqrt{5}-1}{4}-(q_2+q_3)\frac{\sqrt{5}+1}{4}$$
$$= \tfrac{1}{4}(a+5\sqrt{5}\,l)$$
$$\sqrt{p}\cos\theta_2 = \tfrac{1}{4}(a-5\sqrt{5}\,l)$$
$$\sqrt{p}\sin\theta_1 = (q_1-q_4)\sin\frac{2\pi}{5}+(q_2-q_3)\sin\frac{4\pi}{5},$$
$$\sqrt{p}\sin\theta_2 = (q_1-q_4)\sin\frac{4\pi}{5}-(q_2-q_3)\sin\frac{2\pi}{5};$$

therefore $p\sin\theta_1\sin\theta_2 = \frac{\sqrt{5}}{4}(c^2-cd-d^2)$ [*v.* § 1 (2)]
$$= \frac{5\sqrt{5}}{4}(m^2+mn-n^2).$$

By these relations the equation (1) reduces to
$$Y^5-10pY^3-5paY^2+5p\left\{p-\tfrac{1}{4}(a^2-125l^2)\right\}Y$$
$$-p\left\{\frac{a}{8}(a^2-625l^2)-ap-\frac{625}{2}lmn\right\} = 0. \quad (4)$$

We may now return to our supposition that numerical solutions for a, l, m, n have been found in the equations § 1 (3). In applying them to (4), it will readily be seen that a certain ambiguity arises as to the proper sign to be taken for a and lmn. The choice, however, can be easily determined from the following considerations. Since
$$a = 5q_0+1 \equiv 1, \quad \text{mod } 5,$$
there is no ambiguity in its value; and, if (l, m, n) is a correct solution of § 1 (3), the only other sets of solutions are $(l, -m, -n)$, $(-l; n, -m)$, and $(-l, -n, m)$, all of which yield the same product lmn, so that (4) is not altered by these alternative choices.

For instance, $16\times 331 = 61^2+125\,.\,1^2+50\,(5^2+2^2)$,

where $61 = 5^2+4\,.\,10-4$,

so that $a = 61 \equiv 1, \quad \text{mod } 5.$

Let then $l = 1, \quad m = 5, \quad n = -2,$

so that $lmn = -10;$

we see by the above reasoning that all other possible choices of values for l, m, n would have given the same value for the product.

3. It may be noticed that the representation of $16p$ in § 1 (3) depends upon the fact that p, a rational quantity, is found to be the product of two irrational factors $q\omega \,.\, q\omega^4$, which in general, for unrestricted values of the q's, would involve terms containing the square root of 5.

The result may be generalized by supposing a case in which some rational quantity can be expressed as the product of two linear functions of a quadrisectional equation.

Let n be any prime of the form $4k+1$ whose quadrisectional roots are

$$\begin{aligned}\eta_0 &= r \;\;+r^{g^4}+r^{g^8}\;+\ldots,\\ \eta_1 &= r^{g}\;+r^{g^5}+r^{g^9}\;+\ldots,\\ \eta_2 &= r^{g^2}+r^{g^6}+r^{g^{10}}+\ldots,\\ \eta_3 &= r^{g^3}+r^{g^7}+r^{g^{11}}+\ldots,\end{aligned}$$

where $$r = e^{(2\pi/n)\,i},$$

and g is a primitive root of n; and let us suppose that

$$P = (s_0\eta_0+s_1\eta_1+s_2\eta_2+s_3\eta_3)(s_0\eta_2+s_1\eta_3+s_2\eta_0+s_3\eta_1)$$

and the s's are rational.

Let $\eta_0+\eta_1 i^{m}+\eta_2 i^{2m}+\eta_3 i^{3m}$ be written Fi^m, where

$$m \equiv 0,\ 1,\ 2,\ 3, \quad \text{mod } 4;$$

so that $$F1 = -1,$$

and $$\begin{aligned}4\eta_0 &= -1+\;\;Fi+Fi^2+Fi^3,\\ 4\eta_1 &= -1-iFi-Fi^2+Fi^3,\\ 4\eta_2 &= -1-\;\;Fi+Fi^2-Fi^3,\\ 4\eta_3 &= -1+iFi-Fi^2-Fi^3.\end{aligned}$$

Moreover, if $s_0+s_1 i^{m}+s_2 i^{2m}+s_3 i^{3m}$ be written si^m, where

$$m \equiv 0,\ 1,\ 2,\ 3, \quad \text{mod } 4,$$

we have

$$\begin{aligned}16P &= 16\,(s_0\eta_0+s_1\eta_1+s_2\eta_2+s_3\eta_3)(s_0\eta_2+s_1\eta_3+s_3\eta_0+s_3\eta_1)\\ &= (-s1+si^3Fi+si^2Fi^2+siFi^3)(-s1-si^3Fi+si^2Fi^2-siFi^3)\\ &= (-s1+si^2Fi^2)^2-(si^3Fi+siFi^3)^2.\end{aligned}$$

i is the $(\omega^{\frac{1}{2}(n-1)}, g)$ of Bachmann's *Kreistheilungslehre*, p. 84, and

$$Fi^2 = \eta_0+\eta_2-\eta_1-\eta_3 = \sqrt{p},$$

$$(Fi)^2 = (a+bi)\,Fi^2,$$

·e

$$a^2+b^2 = p,$$

$$Fi\,.\,Fi^3 = (-1)^{\frac{1}{2}(n-1)}\,p.$$

nce $16P = (-s1+si^2\sqrt{p})^2-(si^3)^2\,(a+bi)\,\sqrt{p}-(si)^2\,(a-bi)\,\sqrt{p}$
$\qquad -si^3\,.\,si\,(-1)^{\frac{1}{2}(n-1)}\,p.$

quating rational and irrational terms, we have

$$16P = (s_0+s_1+s_2+s_3)^2+n\,(s_0-s_1+s_2-s_3)^2 -2n\,(-1)^{\frac{1}{2}(n-1)}\left\{(s_0-s_2)^2+(s_1-s_3)^2\right\},$$

while

$$(s_0+s_1+s_2+s_3)(s_0-s_1+s_2-s_3) = a\left\{(s_1-s_3)^2-(s_0-s_2)^2\right\}-2b\,(s_1-s_3)(s_0-s_2).$$

These equations may be written in the form

$$\left.\begin{aligned} 16P &= A^2+nB^2-2n\,(-1)^{\frac{1}{2}(n-1)}\,(C^2+D^2) \\ AB &= a\,(D^2-C^2)-2bCD \end{aligned}\right\}. \qquad (1)$$

Two cases of these relations obviously present themselves.

Firstly, we may take the product

$$(x-r\ )(x-r^{s^4})(x-r^{s^8})\,\dots$$
$$\times\,(x-r^{s})(x-r^{s^5})(x-r^{s^9})\,\dots,$$

which can be written in the form

$$s_0\eta_0+s_1\eta_1+s_2\eta_2+s_3\eta_3,$$

while

$$(x-r^{s^2})(x-r^{s^6})\,\dots$$
$$\times\,(x-r^{s^3})(x-r^{s^7})\,\dots$$

becomes

$$s_0\eta_2+s_1\eta_3+s_2\eta_0+s_3\eta_1,$$

where s_0, s_1, s_2, s_3 are rational integral functions of x. $16\,(x^n-1)/(x-1)$ can be written as in (1). For instance,

$$16\,(x^{13}-1)/(x-1) = (4x^6+2x^5-5x^4-2x^3-5x^2+2x+4)^2$$
$$+\,13x^4\,(x+1)^4+26\left\{(x^5-x)^2+(x^5+x^4-\right.$$

$$13 = 3^2+2^2,$$

$$16(x^{17}-1)/(x-1) = (4x^8+2x^7+10x^6-3x^5+8x^4-3x^3+10x^2+2x+4)^2$$
$$+17(2x^6-x^5-x^3+2x^2)^2$$
$$-34\{(x^7+x^5-2x^4+x^3+x)^2+(x^7+2x^5-2x^4+2x^3+x)^2\},$$

while $$17 = 1^2+4^2.$$

Secondly, we may apply (1) to the theory of n-sectional equations in general, where $n = 4k+1$ and is prime, and more especially when $n \equiv 5$, mod 8. The results will have a close analogy to the theorems contained in chapter xx. of Bachmann's *Kreistheilungslehre*, though it is not easy to see how far the result of p. 290 *et seq.* can be generalized.

For instance, when $n \equiv 5$, mod 8, and P is a prime of form $kn+1$, we shall obtain a relation similar to (19), p. 287, viz.,

$$16P^{\frac{1}{4}(n-1)} = A^2+nB^2+2n(C^2+D^2),$$

where $$AB = a(D^2-C^2)-2bCD,$$

and it is probable that in all cases the numbers A, B, C, D contain as a common factor some power of P, reducing the identity to one analogous to (24) on p. 290, the representation being unique.

Again, in what we may call the two-square representation of primes or their powers treated of in this portion of Bachmann's work, the primes of form $4k+1$ are omitted as leading to forms which are ambiguous in consequence of the Pellian equation

$$1 = x^2-ny^2.$$

In the present four-square representation, we can include some such primes unambiguously, viz., those which $\equiv 5$, mod 8, but shall probably meet with a new Pellian ambiguity when the prime $\equiv 1$, mod 8, depending on possible integral solutions of

$$1 = A^2+nB^2-2n(C^2+D^2),$$

where $$AB = a(D^2-C^2)-2bCD.$$

It would be interesting to know how far the integral solution of these equations is possible, both when $n = a^2+b^2$ is prime and when it is composite, and whether any algorithm similar to continued fractions can be employed to determine such a solution.

On a Peculiarity of the Wave-System due to the Free Vibrations of a Nucleus in an Extended Medium. *By* Prof. HORACE LAMB, F.R.S. Read May 10th, 1900. Received September 21st, 1900.

In a number of physical problems we have to do with a central body, or nucleus (usually taken to be spherical for mathematical simplicity), surrounded by a medium capable of transmitting energy in the form of waves. If an agitation be communicated to the nucleus in any way, waves will be started in the medium, and, owing to the energy which these carry off, the vibrations of the nucleus, if this be left to itself, will gradually subside. As instances, we may cite the electrical oscillations on a spherical conductor,* or in a dielectric sphere of great inductive capacity,† and the vibrations of a pendulum or of a deformable sphere in a gaseous‡ or an elastic-solid§ medium. The point to be here considered arises in the interpretation of the analytical expression for the waves thus generated. Though simple, it appears to have occasioned some perplexity, and it has been suggested that a brief elucidation of the matter might be acceptable.

In all problems of the above kind the usual method of procedure is to assume a time-factor e^{ikct}, c being the wave-velocity, and consequently $2\pi/k$ the wave-length (when k is real). The solution of the differential equations which hold throughout the medium then indicates that the amplitude of the disturbance at a distance r from the nucleus, great in comparison with the dimensions of the latter, is given by one or more terms of the type

$$\frac{A}{r}e^{ik(ct-r)}S_n, \tag{1}$$

where S_n involves the direction (only) of the radius vector r. The combination $ct+r$ would of course be equally admissible in the index, but is rejected for our present purpose as representing a system of waves travelling inwards. The admissible values of k are determined in each case by means of the special conditions to be satisfied at the boundary of the nucleus. It is found that k is either pure imaginary, or else complex with the imaginary part positive. In the latter case, writing

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* J. J. Thomson, *Proc. Lond. Math. Soc.*, Vol. xv., p. 197 (1884); *Recent Researches*, § 308.

† *Camb. Trans.*, Vol. XVIII., p. 348.

‡ *Proc. Lond. Math. Soc.*, Vol. XXXII., p. 11.

§ *Ibid.*, Vol. XXXII., p. 120.

$$k = \kappa + im, \tag{2}$$

and substituting in (1), we obtain, on taking the real part, an expression of the form

$$\frac{C}{r} e^{-m(ct-r)} \cos \kappa (ct - r + a) \, S_n. \tag{3}$$

The circular function represents a train of simple harmonic waves of length $2\pi/\kappa$ travelling outwards with velocity c; the factor S_n represents the variation of intensity with direction; and the factor $1/r$ represents the attenuation due to spherical divergence. The factor e^{-mct} exhibits the decay of the vibration at any place as the original energy of the nucleus is gradually spent in the generation of waves. The difficulty which has been felt relates to the factor e^{mr}, m being positive. It is true that m is usually small compared with κ, but, however small it may be, the exponential indicates unlimited increase with r, which will ultimately outbalance altogether the weakening due to spherical divergence. For this reason it has been doubted whether a solution such as (3) really corresponds to any physical reality at all.

The explanation is that owing to the finite velocity of wave-propagation there is at any instant, in our unlimited medium, a region not yet reached by the waves, viz., that beyond a spherical surface of radius ct, where t is the time elapsed since the primitive disturbance of the nucleus. Up to the confines of this region the law of amplitude indicated by (3) is strictly applicable; but beyond them everything is as yet quiescent. And within the region occupied by the waves the amplitude at any point P will (except for spherical divergence) be less than that at a point Q further from the centre on the same radius vector in the ratio $e^{-m \cdot PQ}$, for the reason that it represents a disturbance which started later by an interval PQ/c, during which the vibration of the nucleus has been decaying according to the law e^{-mct}. It is clear in fact that the two parts of the exponential in (3) are necessary concomitants.

When k is pure imaginary, the solution (1) takes the form

$$\frac{A}{r}\, e^{-m(ct-r)}. \tag{4}$$

The only novelty is that the vibration which is sooner or later started at any point is "aperiodic."

To reduce the question to its simplest form, we may take a one-dimensional analogy. Let us suppose that we have an infinitely long tense string of uniform density ρ, to which a mass M is attached at the origin ($x = 0$); and let us further suppose that any lateral displacement of M is resisted not only by the tension (T) of the string, but also by springs fastened to fixed supports; and let $2\pi/\sigma$ be the period in which M would oscillate if the string were absent. Everything being at rest to begin with, let us trace the effect of a sudden blow given to M at time $t = 0$. The equation of motion of the string will be of the form

$$\frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2}, \tag{5}$$

where $c^2 = T/\rho$, and that of M

$$M\left(\frac{d^2y_0}{dt^2} + \sigma^2 y_0\right) = -2T\left(\frac{dy}{dx}\right)_0, \tag{6}$$

where the limit of dy/dx, when x approaches 0 from the positive side of the origin, is to be understood. For $x > 0$, the appropriate solution of (5) is of the form

$$y = f\,(ct - x). \tag{7}$$

If we put $M = 2\rho b$, the equation (6) gives

$$f''(u) + \frac{1}{b}f'(u) + \frac{\sigma^2}{c^2}f(u) = 0, \tag{8}$$

to be satisfied *for positive values of u*. Since when $t = 0$ we have $y = 0$ for all values of x, it appears that for all negative values of u we must have $f\,(u) = 0$. The solution of (8) is

$$f(u) = Ce^{-u/2b}\cos\kappa\,(u + \alpha), \tag{9}$$

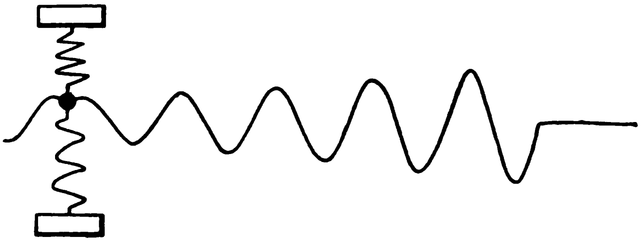
if

$$\kappa = \sqrt{\left(\frac{\sigma^2}{c^2} - \frac{1}{4b^2}\right)}, \tag{10}$$

the constants C and α being (so far) arbitrary. Adjusting α so as to make $f(0) = 0$, since M starts from its equilibrium position, we have

$$\left.\begin{array}{ll} \text{for } x < ct, & y = Ce^{-(ct-x)/2b} \sin \kappa\,(ct-x) \\ \text{and} \quad \text{for } x > ct, & y = 0 \end{array}\right\}. \qquad (11)$$

The formulæ relate to the positive side of the origin. The circumstances are, of course, symmetrical on the two sides. The figure is intended as a rough illustration, the amplitude being of course greatly exaggerated.



We have assumed that $\sigma > c/2b$. In the opposite event, the solution of (8) is

$$f(u) = Ae^{-m_1 u} + Be^{-m_2 u}, \qquad (12)$$

where

$$\left.\begin{array}{l} m_1 \\ m_2 \end{array}\right\} = -\frac{1}{2b} \pm \sqrt{\left(\frac{1}{4b^2} - \frac{\sigma^2}{c^2}\right)}. \qquad (13)$$

Adjusting the constants, we have

$$\left.\begin{array}{ll} \text{for } x < ct, & y = A\left(e^{-m_1(ct-x)} - e^{-m_2(ct-x)}\right) \\ \text{and} \quad \text{for } x > ct, & y = 0 \end{array}\right\}. \qquad (14)$$

Thursday, June 14th, 1900.

Lord KELVIN, G.C.V.O., F.R.S., President, in the Chair.

Twenty-one ordinary members, three honorary members, and a visitor present.

At the request of the President communications were made by Prof. Klein ("On the Continuation of the Edition of Gauss's Collected Works"); by Prof. Darboux ("Sur différents Problèmes relatifs aux

Transformations de l'Espace et aux Déformations finis de la Matière et sur leur rapports avec la Théorie des Systèmes triples orthogonaux); and by Prof. Poincaré ("Sur quelques Théorèmes relatifs à l'Analysis Situs et sur les propriétés des Polyèdres dans l'Espace à plus de trois dimensions"). Professors Darboux and Poincaré were admitted by acclamation, and thanks were voted to all three gentlemen for their communications.

Thanks were also voted to Prof. Stringham, of California, for his remarks on "A Proof by non-Euclidean Geometry of the Focus and Directrix Property of Plane Sections of a Cone."

Prof. Elliott communicated some notes on "Concomitants of Binary Quantics."

Lord Kelvin communicated the following papers by reading out their titles:—

Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications, Part ii., by H. S. Carslaw.

Some Quadrature Formulæ, by W. F. Sheppard.

Extensions of the Riemann-Roch Theorem in Plane Geometry, by Dr. F. S. Macaulay.

On the Invariants of a certain Differential Expression connected with the Theory of Geodesics, by J. E. Campbell.

On the Transitive Groups of degree n and class $n-1$, by Prof. W. Burnside.

The Invariant Syzygies of Lowest Order for any Number of Quartics, by A. Young.

Canonical Reduction of Bilinear Forms, by T. J. I'A. Bromwich.

The Energy Function of a Continuous Medium, by H. M. Macdonald.

Note on the Representation of a Circle by a Linear Equation, by J. Griffiths.

The Stress in an Æolotropic Elastic Solid with an Infinite Plane Boundary, by J. H. Michell.

The following presents were made to the Library:—

"Educational Times," June, 1900.

"Indian Engineering," Vol. xxvii., Nos. 16-20, April 21-May 19, 1900.

"Mittheilungen der Mathematischen Gesellschaft," Bd. iii., Heft 10; Hamburg, 1900.

"*Proceedings* of the American Philosophical Society," Vol. xxxviii., No. 160, *December, 1897; Philadelphia.*

"Mathematical Gazette," Vol. I., No. 21; May, 1900.

"Periodico di Matematica," Serie 2, Vol. II., Fasc. 6; Livorno, 1900.

"Transactions of the American Mathematical Society," Vol. I., No. 2; New York, 1900.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LXVI., No. 429; 1900.

"Rendiconti del Circolo Matematico di Palermo," Tomo XIV., Fasc. 3, 4; 1900.

"Bulletin of the American Mathematical Society," Series 2, Vol. VI., No. 8, May, 1900; New York.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VI., Fasc. 3, 4; Napoli, 1900.

"Journal für die reine und angewandte Mathematik," Band CXXII., Hefte 1, 2; Berlin, 1900.

"Archives Néerlandaises," Serie 2, Tome III., Liv. 5; La Haye, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. IX., Fasc. 8, 9, 10; Roma, 1900.

"Nyt Tidsskrift for Matematik," B. Aargang II., Nr. 2; Copenhagen, 1900.

"Proceedings of the Physical Society," Vol. XVII., Pt. 1; April, 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 1-22; 1900.

"Proceedings of the Cambridge Philosophical Society," Vol. X., Pt. 5 (Lent Term); 1900.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Heft 3, Math.-Physikalische Klasse; 1899.

Notes on Concomitants of Binary Quantics. *By* E. B. ELLIOTT.

Read and received June 14th, 1900. Received, in recast form, July 30th, 1900.

1. Too little attention has, perhaps, been devoted by expositors of the English method to any concomitants of binary quantics except covariants and invariants; the fact that the connexion between cogrediency and contragrediency in the case of pairs of variables is of a very simple nature, and that, for instance, any homogeneous contravariant is made a covariant by a mere substitution for the variables, having been regarded as making the consideration of other concomitants than covariants a matter of secondary interest and importance. Thus, while in respect to a *ternary* quantic it has been made a fundamental

consideration that the system of leading coefficients in concomitants pure and mixed is the system of invariants of an auxiliary binary system, and that the system of leading coefficients in covariants in particular is to be extracted from that system of invariants by means of an additional fact of annihilation by a differential operator, it has not, as it seems to me, been adequately enforced that the answer to the inquiry: What gradients in the coefficients lead concomitants pure and mixed of a *binary* quantic? is: *All gradients.* The elucidation and examination of this fact occupy Section I. of my notes.

Again, it has, perhaps, when noticed, been generally regarded as a mere coincidence that the whole number of seminvariants (including invariants) of any degree i of a binary p-ic is also the number of terms in the most general gradient of degree i and of weight equal to whichever is integral of $\frac{1}{2}ip$ and $\frac{1}{2}(ip-1)$; the fact being that the most general gradient in question, with arbitrary coefficients, may be separated into parts which are coefficients in the covariants led by all those seminvariants, and so have a one-to-one correspondence with them. Facts like this occupy my Sections II. and III. The separation into critical parts of general gradients is effected in Section II. by means of repeated differential operations; and, from any gradient with its full number of arbitraries, an analytical formula is exhibited for that same full number of covariants in which parts of the gradient are coefficients. In Section III. an almost purely algebraical method of effecting the separation is exhibited. The investigations may be regarded as complementary to a theorem of separability given and applied by Hilbert (*Mathematische Annalen*, Vol. xxx., pp. 15, &c.).

A good deal of preliminary exposition of known theory has been unavoidable. The so-called "English," or non-symbolic, method has been followed throughout; the aim of the paper distinctly being to call attention to some neglected lessons of that method and to pursue its development.

I.

2.* Let ξ, η be contragredient with x, y. Take K a rational integral function of x, y; ξ, η; a_0, a_1, a_2, ..., a_p, which is a concomitant of

$$u \equiv (a_0, a_1, a_2, \ldots, a_p)(x, y)^p.$$

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[*] At the suggestion of a referee much of this article has been rewritten, so as to base what follows entirely on the infinitesimal transformation. At his request I have also slightly altered the prefatory article.

The essential properties of K can be written down by expressing its invariancy, but for a factor, when we apply any infinitesimal—and therefore when we apply any—linear transformation. If $1+\lambda$, μ and λ', $1+\mu'$, where λ, μ, λ', μ' are infinitesimal, be the coefficients of the new variables X, Y in the equations of transformation, we express this invariancy by saying that, when we substitute in K

$$(1-\lambda)\,x-\mu y, \quad -\lambda' x+(1-\mu')\,y \quad \text{for} \quad x, y,$$

$$(1+\lambda)\,\xi+\lambda'\eta, \quad \mu\xi+(1+\mu')\,\eta \quad \text{for} \quad \xi, \eta,$$

and $$a_r+\mu r a_{r-1}+\lambda'(p-r)\,a_{r+1}+\left\{\lambda(p-r)+\mu' r\right\}a_r \quad \text{for} \quad a_r$$

$$(r=0, 1, 2, \ldots, p),$$

we reproduce K multiplied by a factor which is, to the first order, of the form $1+b\lambda+b'\lambda'+c\mu+c'\mu'$, where b, b', c, c' are numerical. The necessary and sufficient conditions which follow, the infinitesimals being independent, are, from the coefficients of μ, λ', λ, μ' respectively, that

$$\left(\Omega+\xi\frac{\partial}{\partial\eta}-y\frac{\partial}{\partial x}\right)K=cK,$$

$$\left(O+\eta\frac{\partial}{\partial\xi}-x\frac{\partial}{\partial y}\right)K=b'K,$$

$$\left\{pa_0\frac{\partial}{\partial a_0}+(p-1)\,a_1\frac{\partial}{\partial a_1}+\ldots+a_{p-1}\frac{\partial}{\partial a_{p-1}}-x\frac{\partial}{\partial x}+\xi\frac{\partial}{\partial\xi}\right\}K=bK,$$

$$\left\{a_1\frac{\partial}{\partial a_1}+2a_2\frac{\partial}{\partial a_2}+\ldots+pa_p\frac{\partial}{\partial a_p}-y\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\eta}\right\}K=c'K,$$

where, in Sylvester's notation which I have used elsewhere,

$$\Omega\equiv a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\ldots+pa_{p-1}\frac{\partial}{\partial a_p},$$

$$O\equiv pa_1\frac{\partial}{\partial a_0}+(p-1)\,a_2\frac{\partial}{\partial a_1}+\ldots+a_p\frac{\partial}{\partial a_{p-1}}.$$

The first two conditions can only be satisfied by taking $c=0$, $b'=0$, as the terms of highest weight in K (regarding $x:y$ and $\eta:\xi$ as of weight 1) cannot occur in the left-hand side of the first, or the terms of lowest weight in K in the left-hand side of the second. Thus

$$\Omega+\xi\frac{\partial}{\partial\eta}-y\frac{\partial}{\partial x} \tag{1}$$

and
$$O+\eta\frac{\partial}{\partial\xi}-x\frac{\partial}{\partial y} \tag{2}$$

annihilate K. Hence also their alternant

$$\left(\Omega+\xi\frac{\partial}{\partial\eta}-y\frac{\partial}{\partial x}\right)\left(O+\eta\frac{\partial}{\partial\xi}-x\frac{\partial}{\partial y}\right)$$
$$-\left(O+\eta\frac{\partial}{\partial\xi}-x\frac{\partial}{\partial y}\right)\left(\Omega+\xi\frac{\partial}{\partial\eta}-y\frac{\partial}{\partial x}\right),$$

i.e., the operator

$$\left\{pa_0\frac{\partial}{\partial a_0}+(p-2)\,a_1\frac{\partial}{\partial a_1}+\ldots-pa_p\frac{\partial}{\partial a_p}\right\}$$
$$+\left(\xi\frac{\partial}{\partial\xi}-\eta\frac{\partial}{\partial\eta}\right)-\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right), \tag{3}$$

annihilates K. But this is the difference of the two operators whose effects on K as above are to multiply it by b and c' respectively. Hence $c'=b$; and the factor in the expression of invariancy for the infinitesimal transformation is $1+b\,(\lambda+\mu')$, *i.e.*, is a power of the modulus $1+\lambda+\mu'$. As the product of the determinants of a succession of linear substitutions is the determinant of the resultant substitution, the factor is still the bth power of the modulus in the expression of invariancy for any finite linear transformation.

Thus the necessary and sufficient conditions that K be a concomitant are that it be annihilated by (1) and (2), and therefore by (3)—a group of three operators of infinitesimal transformation—and that

$$a_1\frac{\partial}{\partial a_1}+2a_2\frac{\partial}{\partial a_2}+\ldots+pa_p\frac{\partial}{\partial a_p}-y\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\eta} \tag{4}$$

have the effect on it of a numerical multiplier b.

Let $C_{rs\rho\sigma}$ be the coefficient of $x^r y^s \xi^\rho \eta^\sigma$ in K. The annihilation by (3) tells us that the degree i and weight w of every term in $C_{rs\rho\sigma}$ are connected with the indices by the relation

$$ip-2w+\rho-\sigma-r+s=0, \tag{5}$$

and the fact as to (4) tells us that

$$w+\sigma-s=b=\text{const.} \tag{6}$$

In particular if, as we shall see must be the case, there is a part of K free from y and η, then, for every term $C_{iw}x^{\varpi}\xi^{\varpi'}$ of such part,

$$\begin{aligned} w &= b \\ &= ip-w+\varpi'-\varpi \\ &= \tfrac{1}{2}(ip+\varpi'-\varpi). \end{aligned} \tag{7}$$

This constant is, we have seen, the index of the power of the modulus which occurs in the general expression of invariancy of K.

We now introduce appropriate conventions in supplement to the above necessities. Denote by degree, order, and class respectively dimensions in $a_0, a_1, a_2, \ldots a_p$, in x, y, and in ξ, η. Linear transformation leaves degree, order, and class unaltered. If K contains parts of different degrees, orders, and classes, the transformed parts have in each case their old degree, order, and class. The property of invariancy possessed by K must be possessed by all those parts separately. Though, then, the only limitation on freedom of degree, order, and class which must perforce hold throughout K is

$$p \text{ times degree} + \text{class} - \text{order} = \text{const.}, \tag{8}$$

which is obtained from (5) and (6) by elimination of w, we introduce no real incompleteness of systems of concomitants by conventionally preferring to speak only of concomitants which are of one degree, one order, and one class throughout as simple concomitants, regarding more complex concomitants with the same constant in (8) as linear functions of simple concomitants with the same power of the modulus required in their expressions of invariancy. Henceforward by concomitant we mean, as a rule, " simple " concomitant.

3. Take then a concomitant

$$K \equiv Cx^{\varpi}\xi^{\varpi'}+\ldots \equiv (C, \ldots)(x, y)^{\varpi}(\xi, \eta)^{\varpi'}, \tag{9}$$

which is of one degree i, order ϖ, and class ϖ' throughout.

The leading coefficient C cannot be wanting. For the annihilation of K by (1) tells us among other things that the terms of highest degree in ξ are free from η, and the annihilation by (2) tells us that these terms are annihilated by $O-x\dfrac{\partial}{\partial y}$, so that the part of them which is of highest dimensions in x is free from y. Thus $C \neq 0$.

Moreover, the K led by $Cx^{\varpi}\xi^{\varpi'}$ is unique. Otherwise there would be a concomitant $K_1 - K_2$ with no $x^{\varpi}\xi^{\varpi'}$ term.

C is of one weight throughout by (7), and this weight is given in terms of the other characteristics of K by

$$w = \tfrac{1}{2}(ip + \varpi' - \varpi). \tag{10}$$

We shall presently see that there is no other restriction on the gradient C.

Given C and the order and class, all the other coefficients in K can be derived. In fact, the whole concomitant may be written in either of the forms

$$K \equiv x^{\varpi} e^{(y/x)(O + \eta\,\partial/\partial\xi)}\, \xi^{\varpi'} e^{-\eta/\xi\,\Omega}\, C, \tag{11}$$

$$\equiv \xi^{\varpi'} e^{-(\eta/\xi)(\Omega - y\,\partial/\partial x)}\, x^{\varpi} e^{y/x\,O}\, C. \tag{12}$$

These expressions are afforded by an analysis of the annihilation of K by (1) and (2). The former expresses K as the covariant of u and $\xi x + \eta y$ which has for leading coefficient a seminvariant of u and $\xi x + \eta y$ derived from C, and the latter has the correlative construction. Other forms to which (11) and (12) respectively reduce by Taylor's theorem are

$$K \equiv x^{\varpi - \varpi'} (\xi x + \eta y)^{\varpi'} e^{y/x\,O} e^{-\eta x/(\xi x + \eta y)\,\Omega}\, C, \tag{13}$$

$$\equiv \xi^{\varpi' - \varpi} (\xi x + \eta y)^{\varpi} e^{-\eta/\xi\,\Omega} e^{\xi y\,(\xi x + \eta y)\,O}\, C; \tag{14}$$

two expressions which are known (see my *Algebra of Quantics*, § 99) to be equal when C is any gradient for which (10) holds.

Here, by (10), $\varpi - \varpi' = ip - 2w$. Let ϖ, ϖ' be $ip - w + k$, $w + k$ respectively, where k is zero or a positive or negative integer. Then I have elsewhere shown (*Algebra of Quantics*, *loc. cit.*) that the equivalent forms (13), (14) have for another equivalent

$$K \equiv (\xi x + \eta y)^k\, C(A_0, A_1, A_2, \ldots, A_p), \tag{15}$$

where $(A_0, A_1, \ldots, A_p)(x', y')^p \equiv (a_0, a_1, \ldots, a_p)(xx' - \eta y', yx' + \xi y')^p$, (16)

and $C(A_0, A_1, \ldots, A_p)$ is the same function of A_0, &c., as C is of a_0, &c. [Here k can be negative only when $C(A_0, A_1, \ldots, A_p)$ is divisible by a power of $\xi x + \eta y$.] Thus K can be at once written from $Cx^{\varpi}\xi^{\varpi'}$ either by differential operation or by substitution.

So much when C, ϖ, ϖ' are given.

When an appropriate C alone is given—we defer till the next article a proof that all gradients C of one degree and one weight throughout are appropriate—ϖ and ϖ' are, of course, not arbitrary positive integers. Neither are they determinate, as (10) only gives their difference $\varpi - \varpi' = ip - 2w$. There are smallest admissible positive or zero values of ϖ, ϖ'; but these smallest values may be increased by any, the same, number, and a concomitant K obtained as above.

By (11) or (13) ϖ' must be at least equal to the index of the highest power of Ω which does not annihilate C. For, if $\Omega^{\varpi'+k} C \neq 0$ but $\Omega^{\varpi'+k+1} C = 0$, where $k > 0$, (13) would contain a part which is a numerical multiple of

$$x^{\varpi - \varpi'} (\xi x + \eta y)^{-k} (\eta x)^{\varpi'+k} e^{y/x\,O} \Omega^{\varpi'+k} C,$$

i.e., a part of negative degree in $\xi x + \eta y$, against which no other part could cancel so as to leave the whole integral. In like manner ϖ must be at least equal to the index of the highest power of O which does not annihilate C. These lowest admissible values of ϖ', ϖ are certainly not greater than w and $ip - w$ respectively; but for particular C's they are less. If, for a given appropriate C, $w - k'$ is the smallest value of ϖ' which, with the corresponding

$$\varpi = ip - 2w + \varpi' = ip - w - k',$$

suffices to make K integral, then $ip - w - k'$ is the smallest value of ϖ which, with the corresponding $\varpi' = \varpi - (ip - 2w) = w - k'$, suffices. Moreover, in this case neither $\Omega^{w-k'}$ nor $O^{ip-w-k'}$ can annihilate C. Otherwise, to take the first supposition, (13) would tell us that K has the factor $\xi x + \eta y$, which could be removed, and leave a concomitant $K \div (\xi x + \eta y)$ led by C and of lower class than the lowest possible; and, in like manner, as to the second supposition. We have then

$$\Omega^{w-k'} C \neq 0, \qquad \Omega^{w-k'+1} C = 0,$$
$$O^{ip-w-k'} C \neq 0, \qquad O^{ip-w-k'+1} C = 0.$$

4. We have now to see that every gradient C is appropriate for some value of $\varpi - \varpi'$ or other; in other words, that every rational integral function of the coefficients $a_0, a_1, a_2, \ldots, a_p$ which is of one degree i and one weight w throughout leads a concomitant. This will be proved by showing that the $A_0, A_1, A_2, \ldots, A_p$ of (16) are concomitants, and have for their leading coefficients $a_0, a_1, a_2, \ldots, a_p$ respectively. This will establish that every homogeneous isobaric function of $a_0, a_1, a_2, \ldots, a_p$ leads a concomitant which is that same function of $A_0, A_1, A_2, \ldots, A_p$.

From (16), we have

$$A_0 \equiv (a_0, a_1, \ldots, a_p)(x, y)^p \equiv u, \tag{17}$$

and, for $r = 1, 2, \ldots, p$,

$$A_r \equiv \frac{1}{p(p-1)\ldots(p-r+1)}\left(\xi\frac{\partial}{\partial y} - \eta\frac{\partial}{\partial x}\right)^r u, \tag{18}$$

the leading term in which is $a_r x^{p-r} \xi^r$.

Now these are all concomitants of u, the power of the modulus M in the expression of invariancy of A_r being M^r. For the polarizing operator $\xi\frac{\partial}{\partial y} - \eta\frac{\partial}{\partial x}$ is an invariant operator of modulus M. In fact, from

$$x = lX + mY, \quad y = l'X + m'Y,$$
$$\Xi = l\xi + l'\eta, \quad \mathrm{H} = m\xi + m'\eta,$$

it follows at once that

$$\Xi\frac{\partial}{\partial Y} - \mathrm{H}\frac{\partial}{\partial X} \equiv M\left(\xi\frac{\partial}{\partial y} - \eta\frac{\partial}{\partial x}\right).$$

We may also form the same series of concomitants backwards, starting from the last, the contravariant

$$A_p \equiv (a_p, a_{p-1}, \ldots, a_0)(\xi, -\eta)^p, \tag{19}$$

and producing, for $r = 1, 2, \ldots, p$,

$$A_{p-r} \equiv \frac{1}{p(p-1)\ldots(p-r+1)}\left(y\frac{\partial}{\partial \xi} - x\frac{\partial}{\partial \eta}\right)^r A_p. \tag{20}$$

Hence, if we take any homogeneous rational integral function, of sum of suffixes w and of degree i, of $A_0, A_1, \ldots, A_p$, we obtain a concomitant whose leading term is the same function, of weight w, of $a_0, a_1, \ldots, a_p$ multiplied by $x^{ip-w}\xi^w$, and for which the power of the modulus in the expression of invariancy is M^w.

In particular, by way of example, the cubic

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

has, besides itself, the concomitants

$$(bx^2 + 2cxy + dy^2)\,\xi - (ax^2 + 2bxy + cy^2)\,\eta,$$
$$(cx + dy)\,\xi^2 - 2\,(bx + cy)\,\xi\eta + (ax + by)\,\eta^2,$$
$$d\xi^3 - 3c\xi^2\eta + 3b\xi\eta^2 - a\eta^3,$$

i.e., it has, including itself, concomitants with the leading terms

$$ax^3,\quad bx^2\xi,\quad cx\xi^2,\quad d\xi^3,$$

which may be rationally and integrally compounded so as to produce a concomitant with any homogeneous isobaric function of a, b, c, d for its leading coefficient.

There are then as many linearly independent concomitants of degree i, weight of leading coefficient w, order $ip-w$, and class w, as there are products of weight w of i of the letters $a_0, a_1, a_2, \dots, a_p$. This number is, as usual, written $(w;\, i, p)$.

The number of degree i, weight of leader w, order and class ϖ, ϖ', where $\varpi = ip-w+k$, $\varpi' = w+k$, for a positive k, is the same. We have simply to multiply each by $(\xi x+\eta y)^k$. The set of $(w;\, i, p)$ thus obtained is certainly complete; for the leading coefficients are all linearly independent and as numerous as possible, and no two concomitants have the same leader $Cx^{\varpi}\xi^{\varpi'}$.

Linear functions C of the products of weight w can, as we well know, be so formed that powers of the universal concomitant $\xi x+\eta y$ occur as factors in the derived concomitants of order and class $ip-w$, w. These factors may be removed, and concomitants of order and class

$$\varpi = ip-w-k',\quad \varpi' = w-k'$$

result. In particular, if $ip-w>w$ the covariants of order $ip-2w$, if $ip-w<w$ the contravariants of class $2w-ip$, and if $ip-w=w$ the invariants, are thus obtained.

All concomitants of course, except invariants, become either covariants or zero when in them we replace ξ, η by $y, -x$. The fundamental p-ic concomitants $A_0, A_1, A_2, \dots, A_p$ all of them thus become u itself, but for numerical factors. Those concomitants which become zero on the substitution have $\xi x+\eta y$ for a factor.

5. We have still to consider what are the concomitants of lowest orders and classes which have leading coefficients of weight and degree w, i; in other words, to consider what are the concomitants, with such leaders, that cannot be rationally and integrally expressed in terms of other concomitants and the universal concomitant $\xi x+\eta y$, or in terms of which and $\xi x+\eta y$ all concomitants, with such leaders, can be rationally and integrally expressed.

Take any *covariant*

$$(C_{w'}, C_{w'+1}, \dots, C_{ip-w'})(x, y)^{ip-2w'}, \qquad (21)$$

in which suffixes indicate weight, and in which the degree is i. We will assure ourselves that the coefficients of weight w in such covariants are the leading coefficients of the concomitants in question.

The coefficients in the covariant before us lead concomitants for all of which $ip-2w'$ is the sum $\varpi+\varpi'$ of order and class. In fact, the invariant operator

$$\frac{1}{(ip-2w')(ip-2w'-1)\dots(ip-2w'-r+1)}\left(\xi\frac{\partial}{\partial y}-\eta\frac{\partial}{\partial x}\right)^r$$

$$(r=1,\ 2,\ \dots,\ ip-2w') \quad (22)$$

derives from (21) a series of concomitants, with this sum of order and class, whose leading terms are

$$C_{w'+1}\,x^{ip-2w'-1}\,\xi,$$
$$C_{w'+2}\,x^{ip-2w'-2}\,\xi^2,$$
$$\dots \quad \dots \quad \dots$$
$$C_{ip-w'}\,\xi^{ip-2w'},$$

of which the last is the contravariant derived from the covariant (21) itself by putting ξ, $-\eta$ for y, x, since the concomitant led by $C_{ip-w'}\xi^{ip-2w'}$ must be unique. The general leading term may be written

$$C_{w'+r}\,x^{ip-2(w'+r)+r}\,\xi^r \quad (r=0,\ 1,\ 2,\ \dots,\ ip-2w'). \quad (23)$$

If $C_{w'}$ is the most general seminvariant of its type $(w',\ i)$, these concomitants all involve the same number of arbitraries as does $C_{w'}$. For $C_{ip-w'}$ does, as $C_{w'}$ can be derived from it by repeated operations with Ω, and so do all the intermediaries in consequence or in like manner.

Now alter the notation by putting w for $w'+r$. We see that, if C_w is a coefficient in a covariant whose leading coefficient is C_{w-r}, then

$$C_w\,x^{ip-2w+r}\,\xi^r \quad (24)$$

leads a concomitant. This cannot be divisible by $\xi x+\eta y$, for the substitution in it of y for ξ and $-x$ for η produces not zero, but the covariant led by C_{w-r}. Moreover, no linear function of coefficients C_w in covariants led by different seminvariants C_{w-r}, for the same r, can be equal to a linear function of coefficients C_w in covariants led by seminvariants with smaller r's: for operation with Ω^r would annihilate the right-hand side of any such supposed quantity, but not the left. Thus there are at least as many concomitants (24), with linearly independent leading coefficients of type (w, i), and such that no linear function of those of them of the same order and class, or, therefore, of all of them, is divisible by $\xi x+\eta y$, as there are linearly independent seminvariants of degree i and of all weights $w-r$ between 0 and w or 0 and $ip-w$ inclusive, according as $2w$ is not or is greater than ip. For every one $\varpi-\varpi' = ip-2w$, and, for those with any admissible r, $\varpi+\varpi' = ip-2(w-r)$. Multiplied each by the corresponding zero or positive power $(\xi x+\eta y)^{w-r}$ of the universal concomitant, they produce at least as many linearly independent concomitants for which $\varpi-\varpi' = ip-2w$, $\varpi+\varpi' = ip$ as there are linearly independent seminvariants such as described.

First, suppose that $w \leqq \tfrac{1}{2}ip$, so that $\varpi-\varpi'$ is not negative. We take it as established, by the usual consideration of the annihilation by Ω, that the number of seminvariants of degree i and weight $w-r$ $(r = 0, 1, 2, \ldots, w)$ is not less than

$$(w-r;\ i, p) - (w-r-1;\ i, p).$$

In this case, then, the number of the linearly independent concomitants is not less than a number which is itself not less than

$$\sum_{r=0}^{r=w} \{(w-r;\ i, p)-(w-r-1;\ i, p)\},$$

where $(0; i, p) = 1$, $(-1; i, p) = 0$, *i.e.*, than a number not less than

$$(w;\ i, p).$$

But it certainly cannot exceed this number, which is the full number of arbitraries in the most general gradient of the type (w, i) of the gradients C_w. Hence we draw all the following conclusions—for our present case of $ip \geqq 2w$.

(a) Every concomitant whose leading coefficient is of weight w and degree i has for that leading coefficient either a coefficient C_w in a covariant whose leader is a seminvariant C_{w-r} of weight between 0 and w inclusive, or a sum of such a coefficient C_w and coefficients of weight w in other covariants led by seminvariants of weight greater than $w-r$ but not greater than w. The coefficients C_w themselves lead a complete system of concomitants of degree and weight of leader i, w, and the order and class of the concomitant led by a C_w derived from a seminvariant C_{w-r}, when that concomitant has no factor $\xi x+\eta y$, are $ip-2w+r$ and r respectively. The concomitant whose leader is the most general gradient of type (w, i) consists of the sum of arbitrary multiples of these concomitants given by coefficients C_w, each multiplied by $(\xi x+\eta y)^{w-r}$ for the corresponding value of r assigned by C_{w-r}, and the whole multiplied by an arbitrary power of $\xi x+\eta y$.

(β) The number of gradients C_w which are leading coefficients in concomitants for which $\varpi = ip-2w+r$, and consequently $\varpi' = r$, and no linear function of which has $\xi x+\eta y$ for a factor—or, as is the same thing, the number of asyzygetic coefficients of $x^{ip-2w+r}y^r$ in covariants of order $ip-2(w-r)$—is exactly

$$(w-r;\ i, p)-(w-r-1;\ i, p).$$

Here r lies between 0 and w inclusive.

(γ) The number of seminvariants of type (w', i) $[w' \leqq w$ and so $\leqq \frac{1}{2}ip]$ is exactly, and not merely not less than,

$$(w';\ i, p)-(w'-1;\ i, p).$$

We have further to consider cases when $w > \frac{1}{2}ip$, so that $\varpi-\varpi'$ is negative, and there are contravariants but not covariants led by particular gradients of the type of C_w. In such cases C_{w-r}, the seminvariant leader of a covariant in which a C_w is a coefficient, must be of weight between 0 and $ip-w$ inclusive. Thus r lies between $2w-ip$ and w inclusive, and the number of linearly independent concomitants with leaders C_w is not less than a number of seminvariants not less than

$$\sum_{r=2w-ip}^{r=w} \left\{(w-r;\ i, p)-(w-r-1;\ i, p)\right\},$$

i.e., than $(ip-w;\ i, p)$,

i.e., than $(w;\ i, p)$,

as in the previously considered cases. This number it also cannot exceed, for reasons as before. The number is then exact; and we draw the conclusions:

(δ) The statement (α) holds equally when $w > \frac{1}{2}ip$, except that the seminvariant leaders C_{w-r}, mentioned in the opening sentence and later, are of weights between 0 and $ip-w$ inclusive, instead of between 0 and w.

(ϵ) When $w > \frac{1}{2}ip$ the statement (β) is replaced by:—The number of gradients C_w which are leading coefficients in concomitants for which $\varpi' = r$, and consequently $\varpi = r-(2w-ip)$, and of which no linear function has $\xi x+\eta y$ for a factor, or, which is the same thing, the number of asyzygetic coefficients of $x^{r-(2w-ip)}y^r$ in covariants of order $ip-2(w-r)$, is exactly

$$(w-r;\ i, p)-(w-r-1;\ i, p).$$

Here r lies between $2w-ip$ and w inclusive.

6. The above has all been stated in terms applicable to concomitants of a single binary quantic u only. But the results for a system of binary quantics

$$u_1 \equiv (a_0, a_1, \ldots, a_{p_1})(x, y)^{p_1},$$
$$u_2 \equiv (b_0, b_1, \ldots, b_{p_2})(x, y)^{p_2},$$
$$\&c., \qquad \&c.,$$

are exactly akin to those detailed. We have only to write in the analytical formulæ $\Sigma\Omega$, ΣO, $\Sigma(ip)$ for Ω, O, ip respectively. The convention introduced to supplement necessities, corresponding to that introduced before (8) above, is that the only concomitants regarded as simple are those which, as well as being of one order and one class throughout, are of one degree i_1 throughout in $a_0, a_1, \ldots, a_{p_1}$, the coefficients in the first quantic; one degree i_2 in $b_0, b_1, \ldots, b_{p_2}$, those in the second quantic; and so on. More complex concomitants are linear functions with numerical coefficients of simple concomitants of different orders, classes, and partial degrees, for which

$$\Sigma(ip)+\varpi'-\varpi,$$

which is double the index of the power of the modulus in the expression of invariancy, has the same value.

As in § 4, every coefficient in every one of the quantics leads a concomitant, for which ϖ' is the weight of the coefficient in question, and $\varpi+\varpi'$ is the order of the quantic in which it occurs. Also there are as many linearly independent concomitants of partial degrees $i_1, i_2, \ldots$, weight of leading coefficient w, order $\Sigma(ip)-w$, and class w, as there are products of weight w of i_1 out of $a_0, a_1, \ldots, a_{p_1}$, i_2 out of $b_0, b_1, \ldots, b_{p_2}$, &c. Moreover, as in § 5, every coefficient of weight w in every covariant of degree i leads a concomitant of *order + class = order of covariant*, which concomitant cannot be reduced to one of lower *order + class* by division by $\xi x+\eta y$; and these concomitants are a complete system for the type (w, i) of leaders, in terms of which and $\xi x+\eta y$, the most general concomitant with the leader of type (w, i) can be rationally and integrally expressed. To supply the necessary modification of wording in conclusions (α) to (ϵ) is so direct that a new statement is unnecessary.

II.

7. We have now before us an interesting aspect of the fact that the number of asyzygetic seminvariants (including invariants if $ip = 2w$) of degree i in the coefficients of a binary p-ic, and of weight not exceeding the smaller of w and $ip-w$, namely

$$(w\,;\ i, p),$$

is also the number of terms with arbitrary multipliers in the most general gradient of type w, i; and in particular that the whole number of seminvariants of degree i (including invariants if there be any), namely

$$\left(\frac{ip}{2}\,;\ i, p\right) \quad\text{or}\quad \left(\frac{ip-1}{2}\,;\ i, p\right),$$

according as ip is even or odd, is equal to the number of terms in the most general gradient of type $\left(\frac{ip}{2}, i\right)$ or $\left(\frac{ip-1}{2}, i\right)$, according to this evenness or oddness. The explanation has been seen to be that every seminvariant of degree i leads a covariant, that every coefficient C_w of weight w which occurs in any such covariant leads a concomitant of a fundamental character, that all these coefficients C_w, for the same w, are linearly independent, and that there are as many of them as there are terms in the most general gradient of degree i and weight w, namely $(w\,;\ i, p)$. Now, if $ip \geqq 2w$, the seminvariants which lead covariants involving coefficients of weight w are those whose weight does not exceed w. [If, on the other hand, $ip < 2w$, they are those whose weight does not exceed $ip-w$, and they are $(ip-w\,;\ i, p)$, or still correctly $(w\,;\ i, p)$, in number.] We will confine attention for the present to cases of $ip \geqq 2w$.

A problem which presents itself for solution is the separation of the general gradient of degree i and weight w $(\leqq \tfrac{1}{2}ip)$, containing $(w\,;\ i, p)$ arbitraries, into its parts containing respectively

$$\begin{gathered}(w\,;\ i, p)-(w-1\,;\ i, p),\\ (w-1\,;\ i, p)-(w-2\,;\ i, p),\\ \&c.,\qquad \&c.,\end{gathered}$$

arbitraries which are leaders, second coefficients, &c., in covariants, *i.e.*, which are seminvariants leading covariants of order $ip-2w$, leaders of concomitants of order $ip-2w+1$ and class 1, those of concomitants of order $ip-2w+2$ and class 2, &c.

A method is afforded by Clebsch's process (*Binäre Formen*, § 7) for the expansion of a form, in two pairs of variables, in powers of the determinant of those pairs, with coefficients in the form of polars of forms in one of the two pairs which are themselves polars. Thus by one of (11) to (15) write down the concomitant K, of order $ip-w$ and class w, which is led by the most general gradient of degree i and weight w. This by Clebsch's method can be expanded in the form

$$\sum_{m=0}^{m=w} \mu_m (\xi x+\eta y)^m \left(\eta\frac{\partial}{\partial x}-\xi\frac{\partial}{\partial y}\right)^{w-m} \left(x\frac{\partial}{\partial \eta}-y\frac{\partial}{\partial \xi}\right)^{w-} \left(\frac{\partial^2}{\partial x\partial \xi}+\frac{\partial^2}{\partial y\partial \eta}\right)^m K.$$

The leading coefficients of the concomitants which are here multiplied by the various powers $(\xi x+\eta y)^m$, for $m=0, 1, 2, \ldots, w$, are the linearly independent parts required of the general gradient which leads K.

But what we seek is to separate the general gradient of given type into its critical parts by means of differential operations with regard to $a_0, a_1, a_2, \ldots, a_p$, and to construct the covariants in which the parts are coefficients.

8. Take G the general gradient of degree i and weight w ($\leq \frac{1}{2}ip$), a sum of $(w;\ i, p)$ arbitrary multiples of coefficients. It is known (*cf. Algebra of Quantics*, § 128) that, because $ip-2w > -2$, ΩG, $\Omega^2 G$, ..., $\Omega^w G$ are all general gradients of their types, and so contain respectively $(w-1;\ i, p)$, $(w-2;\ i, p)$, ..., $(0;\ i, p) = 1$, arbitraries. There are then numbers

$$\begin{gathered}(w;\ i, p)-(w-1;\ i, p),\\ (w-1;\ i, p)-(w-2;\ i, p),\\ \ldots\quad \ldots\quad \ldots\quad \ldots\\ (1;\ i, p)-1=0,\\ 1,\end{gathered}$$

of arbitrary multipliers in parts together constituting G, which parts are respectively annihilated by Ω, by Ω^2 but not by Ω, ..., by Ω^w but not by Ω^{w-1}, and by Ω^{w+1} but not by Ω^w; *i.e.*, G is the sum of the above numbers of arbitrary multiples of sets of terms, and so altogether of $(w;\ i, p)$ sets of terms, which when operated on repeatedly by Ω give eventually seminvariants (each at the stage before the one which effects annihilation). The whole number of sets is the number of terms in G.

Now (*Algebra of Quantics*, §§ 180, 181) the operator

$$1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\frac{\Omega^2 O^3\Omega}{1^2.2^2.3^2}+\dots \tag{25}$$

acting on G produces a sum of arbitrary multiples of all the seminvariants of degree i and weight w; the same operator on ΩG, the most general gradient of type $(w-1, i)$, produces a sum of arbitrary multiples of all the seminvariants of degree i and weight $w-1$; the same on $\Omega^2 G$ those of degree i and weight $w-2$, and so on. The orders of covariants led by seminvariants of the types of $G, \Omega G, \Omega^2 G, \dots$ are $ip-2w,\ ip-2w+2,\ ip-2w+4, \dots$, respectively. Thus, if $\lambda_0, \lambda_1, \lambda_2, \dots$ be any (assigned or arbitrary) numerical quantities,

$$\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\frac{\Omega^2 O^3\Omega}{1^2.2^2.3^2}+\dots\right)(\lambda_0+\lambda_1 x^2\Omega+\lambda_2 x^4\Omega^2+\dots)\,x^{ip-2w}G \tag{26}$$

is a sum of arbitrary multiples of the leading terms of all those covariants (including invariants, if $ip = 2w$) of degree i in which there are coefficients of weight w.

Again, $e^{y/x\,O}$ operating on a leading term produces the whole covariant led. Thus

$$e^{y/x\,O}\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\frac{\Omega^2 O^3\Omega}{1^2.2^2.3^2}+\dots\right)(\lambda_0+\lambda_1 x^2\Omega+\lambda_2 x^4\Omega^2+\dots)\,x^{ip-2w}G \tag{27}$$

is, whatever assigned numbers $\lambda_0, \lambda_1, \lambda_2, \dots$ be, a sum of arbitrary numerical multiples of all the $(w;\ i, p)$ covariants described.

Once more, the term whose coefficient is of weight w in a covariant

$$e^{y/x\,O}\,x^{ip-2w+2r}S,$$

where S is a seminvariant of weight $w-r$, is

$$x^{ip-2w+r}\,y^r\,\frac{1}{r!}\,O^r S.$$

Thus

$$\sum_{r=0}^{r=w}\frac{\lambda_r}{r!}\,O^r\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\dots\right)\Omega^r G \tag{28}$$

is, for any assigned numerical $\lambda_0, \lambda_1, \ldots, \lambda_w$, a general gradient of type (w, i) expressed as a sum of w parts of which the $(r+1)$th is the coefficient of weight w which occurs in the most general covariant of degree i and order $ip-2w+2r$.

G and (28) are both general gradients of the same type (w, i). It occurs as desirable so to assign the values of $\lambda_0, \lambda_1, \ldots, \lambda_w$ that they may be identical. In order so to assign them we notice that, if

$$G \equiv H_0 + H_1 + \ldots + H_w,$$

where H_r $(r = 0, 1, 2, \ldots, w)$ is the coefficient of degree i and weight w in a covariant of order $ip-2w+2r$ led by S_r, then

$$S_r = \frac{1}{(ip-2w+2r)(ip-2w+2r-1)\ldots(ip-2w+r+1)}\Omega^r H_r,$$

and that the operator (25) acting on S_r reproduces it exactly. Also

$$\Omega^r (H_0 + H_1 + \ldots + H_{r-1}) = 0$$

and
$$\Omega^r (H_{r+1} + H_{r+2} + \ldots + H_w)$$

is annihilated by (25), because it is of the form OF (*cf. Algebra of Quantics, loc. cit.*). Thus we secure what is desired by taking, after $\lambda_0 = 1$,

$$\lambda_r = \frac{1}{(ip-2w+2r)\ldots(ip-2w+r+1)} = \frac{(ip-2w+r)!}{(ip-2w+2r)!}.$$

Accordingly,

$$G \equiv \sum_{r=0}^{r=w} \frac{(ip-2w+r)!}{r!\,(ip-2w+2r)!} O^r \left(1 - \frac{O\Omega}{1^2} + \frac{\Omega O^2 \Omega}{1^2.2^2} - \,..\right) \Omega^r G \qquad (29)$$

is what exhibits G as separated into the derived sum of coefficients in covariants.

Perhaps, however, it is best so to choose the λ's that G may be split up into a sum

$$G \equiv G_0 + G_1 + \ldots + G_w,$$

in such a way that G_r multiplied by its appropriate binomial coefficient $\dfrac{(ip-2w+2r)!}{r!\,(ip-2w+r)!}$ is the coefficient of weight w, *i.e.*, the multiplier of $x^{ip-2w+r}y^r$, in the general covariant of order $ip-2w+2r$ occurring in the sum (27). The partial G's are thus C_w's of § 5. In this notation

$$S_r = \frac{1}{r!}\,\Omega^r G_r,$$

so that we have to take $\lambda_r = \dfrac{1}{r!}$;

we thus obtain $$G \equiv \sum_{r=0}^{r=w} \frac{O^r}{1^2.2^2\ldots r^2}\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\ldots\right)\Omega^r G, \qquad (30)$$

as the separation of G into parts as required.

The sum of the covariants with the multiples of parts

$$\frac{(ip-2w+2r)!}{r!\,(ip-2w+r)!}\,G_r$$

for coefficients of weight w in them is thus from (27)

$$e^{y/x\,O}\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\ldots\right)\left(1+\frac{x^2\Omega}{1}+\frac{x^4\Omega^2}{1.2}+\ldots\right)x^{ip-2w}G. \qquad (31)$$

The sum of the covariants with the actual parts of G for coefficients of weight w in them is in like manner written down from (27) by substituting

$$\lambda_r = \frac{(ip-2w+2r)!}{(ip-2w+r)!}, \text{ for } r = 0, 1, 2, \ldots, w.$$

9. The consideration of the extreme case when w is whichever is integral of $\tfrac{1}{2}ip$ and $\tfrac{1}{2}(ip-1)$ is of special interest. It in fact gives us all covariants (including invariants) of degree i in one formula.

First, if ip is even, so that there may be invariants. There are altogether $\left(\dfrac{ip}{2};\, i, p\right)$ covariants and invariants of degree i in the coefficients. Take G the most general gradient of degree i and weight $\dfrac{ip}{2}$. It involves just that same number $\left(\dfrac{ip}{2};\, i, p\right)$ of arbitraries. The sum

$$e^{y/x\,O}\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\ldots\right)\sum_{r=0}^{r=\frac{1}{2}ip}\lambda_r x^{2r}\Omega^r G, \qquad (32)$$

contains, for any chosen non-vanishing numerical values of the λ's, that same number of arbitraries, and is the sum of arbitrary multiples of all the covariants and invariants. In particular, by taking

$$\lambda_r = \frac{r!}{(2r)!} \quad (r = 0, 1, 2, \ldots, w),$$

we thus write down from G the general sum of the covariants and invariants in such a way that G is actually separated into the sum of *the coefficients* of weight $\tfrac{1}{2}ip$, *i.e.*, of the middle coefficients in the separate covariants, &c., composing the sum; the separation of G in question being

$$\sum_{r=0}^{r=\frac{1}{2}ip} O^r \left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\dots\right)\frac{\Omega^r}{(2r)!}G. \tag{33}$$

And again by taking $\lambda_r=\frac{1}{r!}$ the separation of G is into the middle coefficients of the separate covariants of the sum, each divided by the binomial coefficient $\frac{(2r)!}{(r!)^2}$ which is appropriate to its position in its own covariant; *i.e.*, into

$$\sum_{r=0}^{r=\frac{1}{2}ip} O^r \left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\dots\right)\frac{\Omega^r}{1^2.2^2\dots r^2}G. \tag{34}$$

In precisely the same way if ip is odd, so that there are no invariants of degree i, but only covariants of the first and higher odd orders, the full system of covariants of degree i is

$$e^{y/x\,O}\left(1-\frac{O\Omega}{1^2}+\frac{\Omega O^2\Omega}{1^2.2^2}-\dots\right)\sum_{r=0}^{r=\frac{1}{2}(ip-1)}\lambda_r x^{2r+1}\Omega^r G \tag{35}$$

for any assigned non-vanishing values of the λ's; and the system of coefficients of $x^{r+1}y^r$ in such covariants, *i.e.*, of coefficients of weight $\frac{ip-1}{2}$ in covariants of degree i, is

$$\sum_{r=0}^{r=\frac{1}{2}(ip-1)}\frac{\lambda_r}{r!}O^r\left(1-\frac{O\Omega}{1^2}+\dots\right)\Omega^r G. \tag{36}$$

The general gradient G of type $\left(\frac{ip-1}{2},\, i\right)$ is exhibited as the sum of parts which are the actual coefficients in question if we take each

$$\lambda_r=\frac{1}{(2r+1)\,2r\,\dots\,(r+2)},$$

and as the sum of parts equal to the actual coeffieients each divided by its appropriate binomial factor if we take each

$$\lambda_r=\frac{1}{r!}.$$

10. We have so far in this section confined ourselves to weights w which do not exceed $\frac{1}{2}ip$. When, on the other hand, $ip<2w$, the reasoning in § 8 has only to be so slightly modified that repetition is hardly necessary. The general gradient G of weight w and degree i

can still be separated into parts, containing in the aggregate the same number of arbitraries as does G, which are coefficients in linearly independent covariants led by seminvariants of degree i. These seminvariants, instead of ranging in weight from 0 to w inclusive as before, now range from 0 to $ip-w$; and the orders of the covariants, consequently, instead of ranging from ip to $ip-2w$ inclusive, range from ip to $ip-2\,(ip-w)$, *i.e.*, $2w-ip$. In (26) and (27) we have to omit the terms in

$$\lambda_0+\lambda_1 x^2\Omega+\lambda_2 x^4\Omega^2+\ldots,$$

which precede $$\lambda_{2w-ip}x^{2(2w-ip)}\Omega^{2w-ip};$$

in (31) to omit the corresponding terms; and in (28), (29) and (30) to replace the range of summation by

$$\sum_{r=2w-ip}^{r=w}.$$

With these modifications the conclusions are as before.

11. The reasoning and results of this section apply equally to covariants and invariants of a set of binary quantics u_1, u_2, u_3, ... of orders p_1, p_2, p_3, ... We have merely to write $\Sigma\Omega$, ΣO and $i_1p_1+i_2p_2+i_3p_3+\ldots$ for Ω, O and ip, where i_1, i_2, i_3, ... are partial degrees in the coefficients of the quantics separately. We thus find, for instance, as in § 9, from the most general gradient of partial degrees i_1, i_2, i_3, ..., and of weight $\frac{1}{2}\Sigma\, ip$, in cases where this is integral, an expression for the sum of arbitrary multiples of all the covariants, including invariants, whose partial degrees are i_1, i_2, i_3, ..., and effect the separation of the general gradient, of the given partial degrees and whole weight, into parts which are the general invariants, and the middle coefficients of the general covariants, of these partial degrees and of the different possible orders from O to $\Sigma\, ip$ inclusive.

III.

12. We have exhibited compact analytical expressions for the separate parts of the general gradient of weight w and degree i of a binary p-ic which are coefficients in the general covariants of degree i and the various different orders. A desideratum which still presents itself is the algebraic performance of the separation without often-repeated application of the operators Ω, O. We shall see that one

operation with each, followed by the solution of algebraic equations, suffices.

The fact which suffices to effect the separation is that, if F is a coefficient in a covariant, $O\Omega F$ is a numerical multiple of F. We shall see, in fact, that if, for a numerical or vanishing k, such an equation as

$$O\Omega F = kF \tag{37}$$

has gradient solutions, then any such solution F is a coefficient in a covariant, and conversely that all coefficients in covariants are given and can be discriminated by means of equations of the form (37).

Suppose that F is a gradient (rational, integral, homogeneous, isobaric function) of degree i and weight w which satisfies an equation (37). We will first see that for this to be possible k must be either a positive integer or zero. Write, as does Sylvester, η for $ip-2w$, which is zero or integral, but is now allowed to be either positive or negative.

It is a known fact (*Algebra of Quantics*, § 125) that, F being any gradient for which $ip-2w = \eta$, and r being a positive integer,

$$O^r\Omega^r F \equiv O\Omega\,(O\Omega-1.\overline{\eta+2})(O\Omega-2.\overline{\eta+3})\,\dots\,(O\Omega-\overline{r-1}.\overline{\eta+r})\,F.$$

If then F satisfies (37), where k is numerical, it follows that

$$O^r\Omega^r F \equiv k\,(k-1.\overline{\eta+2})(k-2.\overline{\eta+3})\,\dots\,(k-\overline{r-1}.\overline{\eta+r})\,F. \tag{38}$$

But for some value of r, and all greater values, $\Omega^r F$, and therefore $O^r\Omega^r F$, vanishes. This must at any rate be the case for $r = w+1$. One at least of the numerical factors on the right must then vanish when $r = w+1$. Let the first which vanishes be the $(t+1)$th, where $w \geqq t \geqq 0$. Then

$$k = t\,(\eta+t+1), \tag{39}$$

a product of two integers, of which the first may be zero, but the second, as we shall see, cannot be. To be sure that k (if not zero) is positive we must assure ourselves that $\eta+t+1$ is positive even when η is not.

To establish this, notice that $O\Omega F = kF$ together with the known identity

$$(\Omega O - O\Omega)\,F = \eta F$$

necessitates
$$\Omega O F = (\eta+k)\,F.$$

Thus there is a companion identity to (38) above; viz.,

$$\Omega^r O^r F = (\eta+k)(\eta+k-1.\overline{2-\eta})(\eta+k-2.\overline{3-\eta})\ \ldots$$

$$\ldots\ (\eta+k-\overline{r-1}.\overline{r-\eta})\ F, \tag{40}$$

and it is necessitated as above that there is a positive or zero t', such that

$$\eta+k = t'(-\eta+t'+1). \tag{41}$$

From this and (39) we obtain, by elimination of k,

$$(t'+t+1)(\eta+t-t') = 0.$$

Here $t'+t+1$ is not zero. Consequently

$$\eta+t = t',$$

which is not negative. Accordingly $\eta+t+1$ is a positive integer; and therefore by (39) so is k, unless $t = 0$, in which case $k = 0$.

Having thus limited the possible values of k in (37), we have now to see that any gradient solution of the equation, with a proper k, is a coefficient in a covariant of our binary quantic.

First suppose that η is positive or zero. We have

$$O^{t+1}\Omega^{t+1}F = 0.$$

Now, for $O^\lambda\Omega^{t+1}F$, $ip-2w'$ is $\eta+2(t+1)-2\lambda$, which is positive if $\lambda < t+1$. Hence, since O cannot annihilate any gradient for which $ip-2w' > 0$, it follows in succession from

$$O^{t+1}\Omega^{t+1}F = 0,$$

i.e.,

$$O.O^t\Omega^{t+}\ F = 0,$$

that

$$O^t\Omega^{t+1}F = 0, \quad O^{t-1}\Omega^{t+1}F = 0, \quad \ldots, \quad \Omega^{t+}\ F = 0,$$

of which facts the last tells us that $\Omega^t F$ is a seminvariant. It is not zero, for $O^t\Omega^t F$ is not. The covariant which it leads is of order

$$ip-2(w-t) = \eta+2t.$$

Also, since

$$k-(s-1)(\eta+s) = t(\eta+t+1)-(s-1)(\eta+s)$$

$$= (t-\overline{s-1})(\eta+t+s),$$

we have from (38), by taking $s = 1, 2, 3, \ldots, t$,

$$O^t\Omega^t F \equiv t(t-1)(t-2)\ \ldots\ 1\ (\eta+t+1)(\eta+t+2)\ \ldots\ (\eta+2t)\ F,$$

i.e.,

$$F \equiv \frac{O^t}{t!}\ \frac{\Omega^t F}{(\eta+t+1)\ \ldots\ (\eta+2t)},$$

so that F is the coefficient of $x^{\eta+t}y^{t}$ in the covariant whose leading term is

$$\frac{\Omega^{t}F}{(\eta+t+1)\ldots(\eta+2t)}\,x^{\eta+2t}. \tag{42}$$

If t is zero, this does not apply verbally, but F is then annihilated by Ω, *i.e.*, is a seminvariant, or first coefficient in a covariant led by Fx^{η}.

We have still to consider cases of η negative. It is best then to reason from (40). We have

$$\Omega^{t'+1}O^{t'+1}F = 0,$$

where $$t' = \eta+t.$$

Now for $\Omega^{\lambda'}O^{t'+1}F$, with $\lambda' < t'+1$, we have

$$ip-2w' = \eta-2\,(t'+1-\lambda') < \eta < 0.$$

Hence, by reasoning as before,

$$O^{t'+1}F = 0,$$

i.e., $O^{t'}F$ is an antiseminvariant, the last coefficient in a covariant whose order is $-\eta+2t'$, *i.e.*, $\eta+2t$; and we obtain

$$\begin{aligned}\eta+k-(s-1)(s-\eta) &= t'\,(-\eta+t'+1)-(s-1)(s-\eta)\\ &= (t'-s-1)(-\eta+t'+s),\end{aligned}$$

and $$F = \frac{\Omega^{t'}}{t'!}\,\frac{O^{t'}F}{(-\eta+t'+1)\ldots(-\eta+2t')},$$

so that F is the coefficient of $x^{t'}y^{-\eta+t'}$, *i.e.*, as before of $x^{\eta+t}\,y^{t}$, in a covariant whose last term is

$$\frac{O^{t'}F}{(-\eta+t'+1)\ldots(-\eta+2t')}\,y^{-\eta+2t'}, \tag{43}$$

where the order $-\eta+2t' = \eta+2t$ as before.

We notice that the two factors

$$t \quad\text{and}\quad \eta+t+1,$$

as the product of which k is exhibited, have for their sum

$$\eta+2t+1,$$

i.e., one more than the order ϖ of the covariant in which F, determined by

$$O\Omega F = kF,$$

is a coefficient, and for their difference

$$\eta+1 \quad\text{or}\quad -\eta-1,$$

according as η is not or is negative.

13. Having now found the forms of k for some or all of which the equation

$$O\Omega F = kF \tag{37}$$

has gradient solutions, and having shown that all solutions which exist are coefficients in covariants of determinate orders, we still have to see that every coefficient in every covariant is thus given. This will be established by showing that every such coefficient satisfies an equation (37).

If $$Sx^{\varpi} + \ldots + Px^{\varpi - t + 1}y^{t-1} + Qx^{\varpi - t}y^{t} + \ldots + S'y^{\varpi}$$

be any covariant, P, Q may be written as

$$\frac{\Omega^{\varpi - t + 1}}{(\varpi - t + 1)!}S', \quad \frac{\Omega^{\varpi - t}}{(\varpi - t)!}S',$$

so that $$\Omega Q = (\varpi - t + 1)\,P,$$

and also as $$\frac{O^{t-1}}{(t-1)!}S, \quad \frac{O^{t}}{t!}S,$$

so that $$OP = tQ.$$

Therefore $$O\Omega Q = t\,(\varpi - t + 1)\,Q,$$

where $$t + (\varpi - t + 1) = \varpi + 1,$$

and $$\begin{aligned} t \sim (\varpi - t + 1) &= \varpi + 1 \sim 2t \\ &= ip - 2w' + 1 \sim 2t, \text{ where } w' \text{ refers to } S, \\ &= 1 \sim (2w - ip), \text{ where } w \text{ refers to } Q. \end{aligned}$$

This difference is $\eta + 1$ or $-\eta - 1$ according as $\eta = ip - 2w$ is not or is negative; which is in full accordance with the last article.

The method used does not apply to first coefficients S, *i.e.*, to seminvariants. But, as $O\Omega S = 0$, there is, with $t = 0$, still accordance as before.

14. We are in position, then, to find all coefficients in all covariants. We seek those of which the degree is i and the weight w, a number which for any gradient lies between 0 and ip inclusive. Write down the most general gradient F of weight w and degree i. It involves $(w;\ i,\ p)$ arbitraries λ. Seek to identify

$$O\Omega F \text{ with } kF.$$

The identification requires the satisfaction of $(w;\ i, p)$ linear homogeneous equations in the $(w;\ i,\ p)$ λ's, k being involved to the first degree in one coefficient in each. To make them consistent we have to choose k so as to satisfy an equation of degree $(w;\ i, p)$ which presents itself in the characteristic determinant form

$$\begin{vmatrix} a_{11}-k, & a_{12}, & a_{13}, & \dots \\ a_{21}, & a_{22}-k, & a_{23}, & \dots \\ a_{31}, & a_{32}, & a_{33}-k, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0. \qquad (44)$$

The solution of this equation of, it may be, high degree is facilitated by our knowledge, from theory adduced above, that its roots must be all real and not fractional or negative. In fact, we know numbers to one or other of which every root must be equal. Thus, if η, *i.e.*, $ip-2w$, is positive (or zero), the roots must consist of

$(w;\ i, p)-(w-1;\ i, p)$ which are zero, equal to $0\,(\eta+1)$ say,

$(w-1;\ i, p)-(w-2;\ i, p)$ which are equal to $1\,(\eta+2)$,

$(w-2;\ i, p)-(w-3;\ i, p)$ „ „ $2\,(\eta+3)$,

&c., &c.; and, lastly,

$(0;\ i, p)$, *i.e.*, one, which is equal to $w\,(\eta+w+1)$.

If on the other hand η is negative (or zero), they consist of

$(w;\ i, p)-(w+1;\ i, p)$ which are equal to $-\eta\,.\,1$,

$(w+1;\ i, p)-(w+2;\ i, p)$ „ „ $(1-\eta)\,2$,

&c., &c.; and, lastly,

$(ip;\ i, p)$, *i.e.*, one, which is equal to $(ip-w-\eta)(ip-w+1)$,

i.e., to $w\,(\eta+w+1)$.

Some of these numbers of roots of particular values may well be zero.

Take any root of the solved equation for k, say one which occurs r times. Substitute it in the equations for the λ's which it makes consistent. These then suffice to determine all the λ's in terms of r of them; and these, on substitution in the gradient F, determine the gradient of greatest generality of the type (w, i) which is a coefficient in a covariant of the order related to the root k. If η is positive (or zero), and the root in question is $t\,(\eta+t+1)$, so that

$$r=(w-t\,;\ i,p)-(w-t-1\,;\ i,p),$$

this order ϖ is $\eta+2t$, *i.e.*, is less by one than the sum of the two factors of k. If on the other hand η is negative, and the root in question is $(t'-\eta)(t'+1)$, so that

$$r=(w+t'\,;\ i,p)-(w+t'+1\,;\ i,p),$$

the order is $2t'-\eta$, *i.e.*, is still less by one than the sum of the two factors of k.

15. As an example, let us discriminate all covariants (and invariants) of degree $i=3$ of the quartic, $p=4$, by separating from one another the five gradients, included in

$$\lambda ace+\mu bcd+\nu ad^2+\rho b^2e+\sigma c^3,$$

which are their middle coefficients.

Identification of $O\Omega F$ with kF gives

$$\frac{10\lambda+\mu+6\nu+6\rho}{\lambda}=\frac{16\lambda+28\mu+24\nu+24\rho+24\sigma}{\mu}=\frac{8\lambda+2\mu+12\nu}{\nu}$$

$$=\frac{8\lambda+2\mu+12\rho}{\rho}=\frac{9\mu+18\sigma}{\sigma}=k.$$

Hence comes the equation for k,

$$\begin{vmatrix} 10-k & 1 & 6 & 6 & . \\ 16 & 28-k & 24 & 24 & 24 \\ 8 & 2 & 12-k & . & . \\ 8 & 2 & . & 12-k & . \\ . & 9 & . & . & 18-k \end{vmatrix}=0,$$

whose roots must occur among

$$0,\ \ 1.2,\ \ 2.3,\ \ 3.4,\ \ 4.5,\ \ 5.6,\ \ 6.7,$$

i.e., $\qquad 0,\ \ 2,\ \ 6,\ \ 12,\ \ 20,\ \ 30,\ \ 42,$

and prove to be $\qquad 0,\ \ 6,\ \ 12,\ \ 20,\ \ 42$

each once.

These values of k in turn inserted give the ratios $\lambda : \mu : \nu : \rho : \sigma$ for

(1) an invariant (covariant of order $\eta = 0$)

$$ace + 2bcd - ad^2 - b^2e - c^3,$$

(2) the mid-coefficient in a covariant of order $2+3-1 = 4$,

(3) „ „ „ $3+4-1 = 6$,

(4) „ „ „ $4+5-1 = 8$,

(5) „ „ „ $6+7-1 = 12$;

the last covariant (5) being the cube of the quartic itself.

16. The other aspect of what is effected, in accordance with Section I. of these notes, by the determination of gradient solutions of equations $O\Omega u = ku$, needs only brief mention. To have determined all the $(w;\ i, p)$ linearly independent gradients of degree i and weight w which are coefficients in covariants, is to have determined the leading coefficients of the $(w;\ i, p)$ concomitants of lowest *order* + *class* which have leading coefficients of the type. Thus a found coefficient C_w for which $k = t(\eta + t + 1)$, as well as being the coefficient of $x^{\eta+t}y^t$ in a covariant of order $\eta + 2t$, is the leading coefficient in a reduced concomitant of order and class ϖ, ϖ', where

$$\varpi - \varpi' = \eta,$$

$$\varpi + \varpi' = \eta + 2t,$$

related to the covariant as a polar in the way exhibited in § 5.

The general concomitant whose leader is the general gradient of type (w, i) is the sum of arbitrary multiples of the concomitants thus obtained, each multiplied by a power of $\xi x + \eta y$ sufficient to raise it to the order $ip - w$ and class w, *i.e.*, $(ip - w) - \eta$, and the whole multiplied by an arbitrary power of $\xi x + \eta y$.

On Transitive Groups of degree n and class $n-1$.

By W. Burnside. Received June 6th, 1900, and communicated June 14th, 1900.

A transitive group of degree n and class $n-1$ contains just $n-1$ substitutions which displace all the n symbols. This property is characteristic; for, if the class be less than $n-1$, the number of substitutions which displace all the n symbols is greater than $n-1$. It is natural to seek to determine whether for such a group the $n-1$ substitutions which are regular in the n symbols, with the identical substitution, constitute a sub-group. No general answer to this question has yet been given. M. E. Maillet, in his thesis, *Recherches sur les Substitutions* (1892), and in subsequent memoirs in the *Bulletin of the French Mathematical Society*, has obtained in this connexion a number of interesting results; and, from quite a different point of view, I have obtained (*Theory of Groups*, pp. 141–144) limitations on the order of the group when the degree is given. In the present communication I show that, unless n is greater than the square of the least odd number which is the order of a simple group, a transitive group of degree n and class $n-1$ necessarily contains a transitive self-conjugate sub-group of order and degree n, consisting of the n substitutions in question. The problem is thus directly connected with another, at present unsolved, question in the theory of groups, viz., that of the possible existence of a simple group of odd order. No detailed investigation of a lower limit for the possible order of a simple group, if odd, has hitherto been undertaken; but I have shown (*loc. cit.*, p. 371) that such a limit must exceed 2835, and it is easy to verify that it certainly cannot be less than 9000. Hence, unless n exceeds 81,000,000, a transitive group of degree n and class $n-1$ is here shown to have a self-conjugate sub-group of order and degree n.

1. Let G be a transitive substitution group of degree n and order N, such that the n sub-groups of G each of which leaves one symbol unchanged are all distinct. Let G_0 be the sub-group of G which leaves the symbol a_0 unchanged, and let

$$a_{01},\ a_{02},\ \ldots,\ a_{0m}$$

be a set of the symbols which are interchanged transitively by G_0. Any substitution of G which changes a_0 into a_r must change the set

$$a_{01}, a_{02}, \ldots, a_{0m}, \text{ or } A_0$$

into the set $$a_{r1}, a_{r2}, \ldots, a_{rm}, \text{ or } A_r,$$

this latter set being one, the symbols of which are interchanged transitively by the operations of the sub-group G_r which leaves a_r unchanged. Every substitution of G which changes a_0 into a_r must change the set A_0 into the set A_r; for otherwise A_r could not be a set of symbols which are interchanged transitively by G_r. Suppose now, further, that a linear function of the symbols of the set A_0 exists, other than their sum, which is changed into a multiple of itself by every operation of G_0. The necessary and sufficient conditions for this are that G_0, so far as it affects the symbols of the set A_0, shall be imprimitive in such a way that the imprimitive systems are merely interchanged cyclically by G_0. When these conditions are satisfied, the linear function of $a_{01}, a_{02}, \ldots, a_{0m}$ can, by taking the symbols in a suitable sequence, be written in the form

$$a_{01} + \epsilon a_{02} + \ldots + \epsilon^{m-1} a_{0m} \quad \text{or} \quad \alpha_0,$$

where ϵ is an mth root of unity, not necessarily a primitive root. If

$$\epsilon^{m'} = 1,$$

where m' is equal to or is a factor of m, then $\alpha_0^{m'}$ is invariant for all the substitutions of G_0. If

$$a_{r1} + \epsilon a_{r2} + \ldots + \epsilon^{m-1} a_{rm} = \alpha_r,$$

where again the symbols are taken in a suitable sequence, then every substitution of G which changes a_0 into a_r must change $\alpha_0^{m'}$ into $\alpha_r^{m'}$, and α_0 into $\epsilon' \alpha_r$, where ϵ' is some integral power of ϵ.

Let the N operations T_s ($s = 1, 2, \ldots, N$) of G be

$$a'_r = a_{r_s} \quad (r = 0, 1, \ldots, n-1),$$

where $0_s, 1_s, \ldots, (n-1)_s$ are $0, 1, \ldots, (n-1)$, in some new sequence. The effect of these substitutions on the α's is to give N linear substitutions T'_s ($s = 1, 2, \ldots, N$)

$$\alpha'_r = \epsilon_{r,s} \alpha_{r_s} \quad (r = 0, 1, \ldots, n-1).$$

These linear substitutions constitute a group G' which is simply isomorphic with G, so that T_s and T'_s are corresponding operations.

The quantities $\epsilon_{r,s}$ are m'th roots of unity, and some of them are certainly distinct from unity.

Let T_s be an operation of G whose order is relatively prime to m'. If it permutes the symbols cyclically in sets of $m_1, m_2, \ldots, m_t$, its multiplier equation will be

$$(\lambda^{m_1}-1)(\lambda^{m_2}-1)\ldots(\lambda^{m_t}-1)=0,$$

where $m_1, m_2, \ldots, m_t$ are relatively prime to m'. The multiplier equation of T'_s will be

$$(\lambda^{m_1}-\epsilon_1)(\lambda^{m_2}-\epsilon_2)\ldots(\lambda^{m_t}-\epsilon_t)=0,$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_t$ represent products of the n quantities $\epsilon_{r,s}$ in sets of $m_1, m_2, \ldots, m_t$. Hence, as T'_s is of the same order as T_s, $\epsilon_1, \epsilon_2, \ldots, \epsilon_t$ must all be unity, and therefore

$$\prod_{r=0}^{r=n-1} \epsilon_{r,s} = 1.$$

The product of two operations of G' for which this last equation holds is obviously another operation of G' for which it holds. Now the totality of the operations of G, whose orders are relatively prime to m' generate a self-conjugate sub-group, and, unless the equation

$$\prod_{r=0}^{r=n-1} \epsilon_{r,s} = 1$$

holds for all the operations of G', this sub-group cannot coincide with G itself. Hence, unless the product $\prod_{r=0}^{r=n-1} \epsilon_{r,s}$ is unity for all the operations of G', the group G is composite and has a self-conjugate sub-group constituted of the totality of the operations for which this product is unity.*

2. The result thus obtained is now to be applied to transitive groups of degree n and class $n-1$. The order of such a group is $n\nu$, where ν is a factor of $n-1$, and a sub-group of order ν which leaves one symbol unchanged permutes the remainder regularly in $\frac{n-1}{\nu}$

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* [*Sept. 4th*, 1900.—This result is immediately obvious by considering the effect of the substitutions of G' on the product $a_0 a_1 \ldots a_{n-1}$. Unless $\prod_{n=0}^{r=n-1} \epsilon_{r,s} = 1$ for each substitution, a cyclical group thus arises with which G' and G are multiply *isomorphic.]*

transitive sets of ν each. Hence, if G_0 is not a perfect group, the conditions are satisfied for the existence of a linear invariant a_0 of G_0. With the notation of § 1 m' is a factor of ν; and the order of an operation S' of G', which changes a_0 in ϵa_0, must be some multiple $m'p$ of m'. If S is the corresponding operation of G, then S must leave a_0 unchanged and permute the remaining $n-1$ symbols cyclically in sets of $m'p$ each. Let

$$a_1' = a_2, \quad a_2' = a_3, \quad \ldots, \quad a_{m'p}' = a_1$$

be a cycle of S. Then

$$a_1' = \epsilon_1 a_2, \quad a_2' = \epsilon_2 a_3, \quad \ldots, \quad a_{m'p}' = \epsilon_{m'p} a_1$$

is the corresponding part of S'; and, since $m'p$ is the order of S',

$$\epsilon_1 \epsilon_2 \ldots \epsilon_{m'p} = 1.$$

Hence, for S',

$$\prod_{r=0}^{r=n-1} \epsilon_{r,s} = \epsilon,$$

and therefore G has a self-conjugate sub-group which does not contain S. The order of any operation of G which displaces all the symbols is prime relatively to ν, and therefore to m'. Hence the self-conjugate sub-group for which

$$\prod_{r=0}^{r=n-1} \epsilon_{r,s} = 1$$

contains all the operations of G which displace all the symbols. This self-conjugate sub-group then appears as a transitive group of degree n, class $n-1$, and order $n\nu'$, where $\nu' = \nu/m'$. If the sub-groups of degree ν' are not perfect, the same process may be repeated. Hence, finally, if the sub-group of order ν is soluble, the $n-1$ operations which displace all the symbols form, with the identical operation, a self-conjugate sub-group.

If ν is even, I have shown (*loc. cit.*) that this self-conjugate sub-group always exists, and that it is then Abelian. It is also proved in the same place that, if ν is not less than $\sqrt{n}$, the degree n must be the power of a prime, so that again, in this case, the self-conjugate sub-group of order and degree n necessarily exists. If l is the smallest odd number which is the order of a simple group, every group of odd order less than l is necessarily soluble. Hence, finally, unless n is greater than l^2, a transitive group of degree n and class $n-1$ necessarily contains a self-conjugate sub-group of order and degree n.

3. It has been shown in the preceding section that, if the sub-groups of order ν of a transitive group G of degree n, order $n\nu$, and class $n-1$ are soluble, then G has a transitive self-conjugate sub-group of order n. I proceed now to prove that, if G has such a self-conjugate sub-group, then the sub-groups of order ν are soluble, with a single possible exception.

Let H be the transitive self-conjugate sub-group of order n of G now assumed to exist; and, $p^{\alpha'}$ being the highest power of a prime p which divides n, let P' be a sub-group of H of order $p^{\alpha'}$. There must be a sub-group K of G of order ν each of whose operations transforms P' into itself; as otherwise G would contain more sub-groups of order $p^{\alpha'}$ than H contains. Hence $\{P', K\}$ is a group of order $p^{\alpha'}\nu$ which contains P' self-conjugately. Let P, of order p^{α}, be that characteristic sub-group of P' which consists of all the self-conjugate operations of P' whose orders are p. Then every operation of K transforms P into itself, and therefore $\{P, K\}$ is a group of order $p^{\alpha}\nu$ which contains P self-conjugately. Moreover, no operation of K is permutable with any operation of P. Hence K can be represented as a group of isomorphisms of P; and no one of these isomorphisms leaves any operation of P, except identity, unchanged.

Let q be any prime factor of ν, and suppose, if possible, that K contains two permutable operations of order q which are not powers of each other. These will generate a group of isomorphisms of P of order q^2. If

$$T_{r,s} \quad (r = 1, 2, \ldots, q\,;\ s = 1, 2, \ldots, q)$$

be a set of q^2 operations of P which are interchanged transitively by this group of isomorphisms, the two generating operations of the group (so far as it affects this set of symbols) may be taken to be

$$(T_{1,1}T_{1,2}\ldots T_{1,q})\ \ldots\ (T_{q,1}T_{q,2}\ldots T_{q,q})$$

and $$(T_{1,1}T_{2,1}\ldots T_{q,1})\ \ldots\ (T_{1,q}T_{2,q}\ldots T_{q,q}),$$

and from these the remaining operations may be at once written down. The product of the T's contained in any one cycle of any of these isomorphisms is an operation of P which is changed into itself. It must therefore be the identical operation. Now in the $q+1$ products thus formed which contain $T_{1,1}$ each of the other T's occurs just once. Hence

$$T_{1,1}^{q+1}\,T_{1,2}\ \ldots\ T_{q,q} = 1.$$

But $$T_{1,1}\,T_{1,2}\ \ldots\ T_{q,q} = 1,$$

since the left-hand side is the product of the T's in the q cycles of any one isomorphism. Hence

$$T^q_{1,1} = 1,$$

which is not true. The supposition that K contains two permutable operations of order q which are not powers of each other therefore leads to a contradiction. Hence, if q^β be the highest power of a prime q which divides ν, a sub-group of K of order q^β has only one sub-group of order q. It follows immediately* that, if q is odd, the sub-groups of K of order q^β are cyclical; while, if q is 2, they are either cyclical or of the type

$$S^{2^{\beta-1}} = 1, \quad \Sigma^2 = S^{2^{\beta-2}}, \quad \Sigma^{-1} S \Sigma = S^{-1}.$$

A group of order $p^\alpha q^\beta \ldots$ in which the sub-groups of order p^α, q^β, ... are cyclical, is always soluble;† and a group of order $2^\alpha q^\beta, \ldots$ in which the sub-groups of order 2^α are of the above type, while those of order q^β, ... are cyclical, is certainly soluble,‡ unless the order is divisible by 3. Hence in a transitive group of degree n, class $n-1$, and order $n\nu$, the sub-groups of order ν are, with a single possible exception, soluble if the group has a sub-group of order n.

4. The main results obtained in this note may be stated, apart from the phraseology of substitution groups, as follows:—

If a group of order nm, where n and m are relatively prime, contains n conjugate sub-groups of order m which have no common operations except identity, and if these sub-groups are soluble, then the group has a self-conjugate sub-group of order n.

If these conditions are satisfied, and m is the power of an odd prime (in which case the sub-groups of order m are necessarily soluble), then these sub-groups are cyclical.

If a group admits a group of isomorphisms whose order is a power of an odd prime, no one of which leaves any operation of the group except identity unchanged, the group of isomorphisms is cyclical.

If a group admits a group of isomorphisms whose order is a power of 2, no one of which leaves any operation of the group unchanged except identity, the group of isomorphisms is either cyclical or of the type defined above.

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* *Theory of Groups*, pp. 72–75.

† *Ibid.*, p. 352.

‡ *Ibid.*, p. 364.

[*Note, July* 10*th*, 1900.—An odd number certainly cannot be the order of a simple group unless it has more than five prime factors. Moreover, it must not be of the forms p^α, $p^\alpha q$, $p^\alpha q^2$, $p^\alpha q^\beta$, ..., r^γ, where, in the last form, each index is either 1 or 2. These results are proved in the last chapter of my *Theory of Groups*.

Now the only odd numbers less than 9000 which satisfy these conditions are :—

(i) $3^5.5.7$, (ii) $3^4.5.7$, (iii) $3^4.5.11$, (iv) $3^4.5.13$,
(v) $3^4.5.17$, (vi) $3^4.5.19$, (vii) $3^4.7.11$, (viii) $3^4.7.13$,
(ix) $3^3.5^3$, (x) $3^3.5^2.7$, (xi) $3^3.5^2.11$, (xii) $3^3.5^2.13$,
(xiii) $3^3.5.7^2$, (xiv) $3^2.5^3.7$.

Groups of orders (i), ..., (vii) may be shown immediately to have a self-conjugate sub-group whose order is a power of 3. A group of order (viii), if simple, could be represented as a substitution group of degree 27, and the sub-group of order 13.7.3, which keeps one symbol unchanged, could be expressed as a group of degree 13; it is known that no such group exists. A group of order (ix) must have a self-conjugate sub-group of order 5^3. A group of order (x) must have a self-conjugate sub-group whose order is a power of 5. A group of order (xi), if simple, could be expressed as a substitution group of degree 11 ; it is known that no such group exists. A group of order (xii) must have a self-conjugate sub-group whose order is a power of 5. A group of order (xiii), if simple, could be expressed as a substitution group of degree 15; no such group exists. A group of order (xiv) must have a self-conjugate sub-group whose order is a power of 5. No group whose order is odd and less than 9000 then can be simple; and every such group is, therefore, soluble. The statement in the introduction is thus justified.]

The Stress in an Æolotropic Elastic Solid with an Infinite Plane Boundary. *By* J. H. Michell.

Received June 7th, 1900. Communicated June 14th, 1900.

In view of the recognized æolotropy of so many materials of construction, it seems to be advisable to extend, as far as possible, to æolotropic solids the general solutions which have been obtained for isotropic solids. The solids considered in the present paper have an axis of elastic symmetry, and are elastically equivalent to crystals of the hexagonal system.

The method employed is similar to that used in a previous paper* on the corresponding solutions for isotropic solids. After putting the general equations into forms suitable for the deduction of potential solutions, the problem of given surface-tractions on an infinite plane boundary which is normal to the axis of symmetry is treated in detail.

The solutions of the other problems treated in the previous paper are obtained in a similar manner, and it has not been thought necessary to reproduce them here.

The older results referred to in the present paper are all given in Todhunter and Pearson's *History of Elasticity*, Vol. II., §§ 1485–1519.

The Volume-Equations.

1. The solid being homogeneous and elastically symmetrical around lines parallel to the axis of z, the energy of deformation W is given by†

$$2W = A(e+f)^2 + 2F(e+f)g + Cg^2 + L(a^2+b^2) + N(c^2-4ef);$$

and therefore the elements of stress by

$$P = A\theta + (F-A)g - 2Nf,$$

$$Q = A\theta + (F-A)g - 2Ne,$$

$$R = F\theta + (C-F)g,$$

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* *Proc. Lond. Math. Soc.*, Vol. XXXI., p. 183.

† A. E. H. Love, *Elasticity*, I., § 38.

$$S = La,$$
$$T = Lb,$$
$$U = Nc,$$

where $$\theta = e+f+g.$$

The equations of motion

$$P_x+U_y+T_z = \rho\ddot{u}-X,$$
$$U_x+Q_y+S_z = \rho\ddot{v}-Y,$$
$$T_x+S_y+R_z = \rho\ddot{w}-Z$$

become

$$A\theta_x+(F-A)\,g_x-2Nf_x+Nc_y+Lb_z = \rho\ddot{u}-X, \tag{1}$$
$$A\theta_y+(F-A)\,g_y-2Ne_y+Nc_x+La_z = \rho\ddot{v}-Y, \tag{2}$$
$$F\theta_z+(C-F)\,g_z+L\,(b_x+a_y) \qquad = \rho\ddot{w}-Z. \tag{3}$$

Now

$$\begin{aligned} b_x+a_y &= (u_z+w_x)_x+(w_y+v_z)_y \\ &= \theta_z+\nabla^2 w-2w_{zz}; \end{aligned}$$

therefore (3) becomes

$$(F+L)\,\theta_z+(C-F-2L)\,w_{zz}+L\nabla^2 w = \rho\ddot{w}-Z. \tag{4}$$

Differentiating (1), (2) with respect to x, y, respectively, and adding, we get, remembering the identical strain-relation

$$c_{xy}-f_{xx}-e_{yy} = 0,$$

$$(L-A)\,\theta_{zz}+A\nabla^2\theta+(F+L-A)\,\nabla^2 w_z-(F+2L-A)\,w_{zzz}$$
$$= \rho\,(\ddot{\theta}-\ddot{w}_z)-\Lambda+Z_z, \tag{5}$$

where $$\Lambda = X_x+Y_y+Z_z.$$

Differentiating (1), (2) with respect to y, x, respectively, and subtracting, we get

$$(L-N)\,\varpi_{zz}+N\nabla^2\varpi = \rho\ddot{\varpi}-\Omega, \tag{6}$$

where

$$2\varpi = v_x-u_y,$$
$$2\Omega = Y_x-X_y,$$

and the identities

$$\begin{aligned} c_x-2e_y &= \quad (v_x-u_y)_x = \quad 2\varpi_x, \\ c_y-2f_x &= -(v_x-u_y)_y = -2\varpi_y, \\ a_x-b_y &= 2\varpi_z \end{aligned}$$

have been used.

Equations (4), (5), (6) are the volume-equations to be employed.

Boundary-Conditions.

2. (*a*) If the stresses R', S', T' are given over $z = 0$, the boundary-conditions are

$$F\theta + (C-F)\, w_z = R', \tag{7}$$

$$La = S', \tag{8}$$

$$Lb = T'. \tag{9}$$

From (8), (9), we get $$2L\varpi_z = S'_{x'} - T'_{y'}, \tag{10}$$

where (x', y') is a point on $z = 0$.

Further, $$L\,(b_x + a_y) = T'_{x'} + S'_{y'},$$

which, on substitution from (3), becomes

$$F\theta_z + (C-F)\, w_{zz} = \rho\ddot{w} - Z - T'_{x'} - S'_{y'}. \tag{11}$$

Equations (7), (10), (11) are the surface-conditions corresponding to the volume-equations (4), (5), (6).

(*b*) If the surface-displacements are given, the conditions are

$$u = u',$$

$$v = v',$$

$$w = w',$$

and from these we derive

$$\left.\begin{aligned} 2\varpi &= v'_{x'} - u'_{y'} \\ \theta - w_z &= u'_{x'} + v'_{y'} \\ w &= w' \end{aligned}\right\}, \tag{12}$$

which are the appropriate surface-conditions in this case.

Symmetrical Form of Volume-Equations of Equilibrium.

3. Suppose now that there are no volume-forces. The equations (4), (5) in the case of equilibrium become

$$(F+L)\,\theta_z + (C-F-2L)\, w_{zz} + L\nabla^2 w = 0,$$

$$(L-A)\,\theta_{zz} + A\nabla^2\theta - (F+2L-A)\, w_{zzz} + (F+L-A)\,\nabla^2 w_z = 0.$$

Differentiate the former with respect to z, multiply by a constant p, and add to the latter. We thus get

$$\nabla^2 \left[A\theta + (F+L-A+pL)\, w_z \right]$$
$$+ \frac{d^2}{dz^2} \left[\{ L-A+p(F+L) \} \theta + \{ p(C-F-2L) - (F+2L-A) \} w_z \right] = 0.$$

Now choose p so that

$$\frac{F+L-A+pL}{A} = \frac{p\,(C-F-2L)-(F+2L-A)}{L-A+p\,(F+L)} = q \text{ (say).}$$

The equation becomes

$$\left\{ \nabla^2 + (\gamma-1) \frac{d^2}{dz^2} \right\} (\theta + q w_z) = 0,$$

where
$$\gamma - 1 = \frac{L-A+p\,(F+L)}{A}$$

or
$$\gamma = \frac{L+p\,(F+L)}{A}.$$

The equation for p is

$$p^2 L\,(F+L) + p\,\{ (F+L)^2 + L^2 - AC \} + L\,(F+L) = 0.$$

The roots are real and unequal if

$$(F^2 - AC)\,\{ (F+2L)^2 - AC \} > 0.$$

Since for the stability of the material $F^2 - AC$ must be negative, the condition becomes

$$(F+2L)^2 < AC.$$

There is no obvious reason why this condition should be satisfied, but this is of no importance, as the method employed is equally applicable to the case of imaginary roots. The case of equal roots is most simply treated as a limiting case of unequal roots, and the solution may, as such, be derived without difficulty from the general solution. For definiteness, the roots will be supposed real and unequal in what follows, and, distinguishing them by suffixes, we have the two equations

$$\left. \begin{aligned} \left\{ \nabla^2 + (\gamma_1 - 1) \frac{d^2}{dz^2} \right\} (\theta + q_1 w_z) &= 0 \\ \left\{ \nabla^2 + (\gamma_2 - 1) \frac{d^2}{dz^2} \right\} (\theta + q_2 w_z) &= 0 \end{aligned} \right\}, \qquad (13)$$

where
$$\gamma_1\gamma_2 = \frac{\{L+p_1(F+L)\}\{L+p_2(F+L)\}}{A^2}$$
$$= \frac{L^2+(F+L)^2-\{(F+L)^2+L^2-AC\}}{A^2}$$
$$= \frac{C}{A}$$

and
$$\gamma_1+\gamma_2 = \frac{2L^2-\{(F+L)^2+L^2-AC\}}{AL}$$
$$= \frac{AC-F^2-2FL}{AL}.$$

Now $AC > (F+2L)^2$ for real roots, and $AC > F^2$ always; therefore $2(AC-F^2-2FL)$ or $AC-F^2+AC-(F+2L)^2+4L^2$ is positive.

The roots γ_1, γ_2 are therefore positive, if real.

If there are volume-forces, the equations (13) become

$$\left.\begin{aligned} A\left\{\nabla^2+(\gamma_1-1)\frac{d^2}{dz^2}\right\}(\theta+q_1w_z) &= -\Lambda+(1-p_1)Z_z \\ A\left\{\nabla^2+(\gamma_2-1)\frac{d^2}{dz^2}\right\}(\theta+q_2w_z) &= -\Lambda+(1-p_2)Z_z \end{aligned}\right\}. \qquad (14)$$

Solution of Equations.

4. We now proceed to find the solution for the case of given stresses on an infinite plane boundary. The solid is supposed to occupy the region in which z is positive and to be fixed at infinity. The boundary-conditions

$$F\theta+(C-F)w_z = R',$$
$$F\theta_z+(C-F)w_{zz} = -T'_x-S'_y$$

may be written

$$\alpha(\theta+q_1w_z)+\beta(\theta+q_2w_z) = R',$$
$$\alpha\frac{d}{dz}(\theta+q_1w_z)+\beta\frac{d}{dz}(\theta+q_2w_z) = -T'_x-S'_y,$$

where
$$\alpha+\beta = F,$$
$$\alpha q_1+\beta q_2 = C-F,$$

so that
$$\alpha(q_1-q_2) = C-F-q_2F,$$
$$\beta(q_2-q_1) = C-F-q_1F.$$

Thus, putting
$$\theta+q_1w_z = V_1,$$
$$\theta+q_2w_z = V_2,$$

we have, to find V_1, V_2, the volume-equations

$$\frac{d^2V_1}{dx^2}+\frac{d^2V_1}{dy^2}+\gamma_1\frac{d^2V_1}{dz^2}=0,$$

$$\frac{d^2V_2}{dx^2}+\frac{d^2V_2}{dy^2}+\gamma_2\frac{d^2V_2}{dz^2}=0,$$

and the boundary-conditions

$$\alpha V_1+\beta V_2 = R',$$

$$\alpha\frac{dV_1}{dz}+\beta\frac{dV_2}{dz} = -T'_{x'}-S'_{y'}.$$

At infinity V_1, V_2 vanish. Let

$$r_1^2 = (x-x')^2+(y-y')^2+z^2/\gamma_1,$$
$$r_2^2 = (x-x')^2+(y-y')^2+z^2/\gamma_2.$$

The equations can be satisfied by putting

$$2\pi V_1 = H_1\iint\frac{\mu}{r_1}\,dx'dy'-K_1\frac{d}{dz}\iint\frac{\nu}{r_1}\,dx'dy',$$

$$2\pi V_2 = H_2\iint\frac{\mu}{r_2}\,dx'dy'-K_2\frac{d}{dz}\iint\frac{\nu}{r_2}\,dx'dy',$$

where μ, ν are functions of x', y', to be determined. These give, at $z = 0$,

$$2\pi V_1 = H_1\iint\frac{\mu}{r}\,dx'dy'+\frac{2\pi K_1\nu}{\sqrt{\gamma_1}},$$

$$2\pi V_2 = H_2\iint\frac{\mu}{r}\,dx'dy'+\frac{2\pi K_2\nu}{\sqrt{\gamma_2}},$$

$$2\pi\frac{dV_1}{dz} = -\frac{2\pi H_1\mu}{\sqrt{\gamma_1}}+\frac{K_1}{\gamma_1}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\iint\frac{\nu}{r}\,dx'dy',$$

$$2\pi\frac{dV_2}{dz} = -\frac{2\pi H_2\mu}{\sqrt{\gamma_2}}+\frac{K_2}{\gamma_2}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\iint\frac{\nu}{r}\,dx'dy',$$

where
$$r^2 = (x-x')^2+(y-y')^2.$$

Hence we must put

$$\alpha H_1+\beta H_2=0$$

$$\frac{\alpha H_1\mu}{\sqrt{\gamma_1}}+\frac{\beta H_2\mu}{\sqrt{\gamma_2}}=T'_{x'}+S$$

$$\frac{\alpha K_1}{\gamma_1}+\frac{\beta K_2}{\gamma_2}=0,$$

$$\frac{\alpha K_1\nu}{\sqrt{\gamma_1}}+\frac{\beta K_2\nu}{\sqrt{\gamma_2}}=R',$$

so that

$$\alpha H_1\mu=-\beta H_2\mu=\frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,(T'_{x'}+S'_{y'})$$

$$\frac{\alpha K_1\nu}{\gamma_1}=-\frac{\beta K_2\nu}{\gamma_2}=-\frac{1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,R'.$$

The required solution for V_1, V_2 is therefore

$$2\pi\alpha V_1=\frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint(T'_{x'}+S'_{y'})\,\frac{1}{r_1}\,dx'dy'$$
$$+\frac{\gamma_1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,\frac{d}{dz}\iint R'\,\frac{1}{r_1}\,dx'dy',$$

$$2\pi\beta V_2=-\frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint(T'_{x'}+S'_{y'})\,\frac{1}{r_2}\,dx'dy'$$
$$-\frac{\gamma_2}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,\frac{d}{dz}\iint R'\,\frac{1}{r_2}\,dx'dy'.$$

Integrating by parts, these may be written

$$\left.\begin{aligned}V_1&=\frac{d\Phi_1}{dx}+\frac{d\Psi_1}{dy}+\frac{d\mathrm{X}_1}{dz}\\ V_2&=\frac{d\Phi_2}{dx}+\frac{d\Psi_2}{dy}+\frac{d\mathrm{X}_2}{dz}\end{aligned}\right\}, \tag{15}$$

where

$$\Phi_1=\frac{1}{2\pi\alpha}\,\frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint T'\,\frac{1}{r_1}\,dx'dy',$$

$$\Psi_1=\frac{1}{2\pi\alpha}\,\frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint S'\,\frac{1}{r_1}\,dx'dy',$$

$$\mathrm{X}_1=\frac{1}{2\pi\alpha}\,\frac{\gamma_1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint R'\,\frac{1}{r_1}\,dx'dy',$$

and Φ_2, Ψ_2, X_2 are obtained by interchanging the suffixes 1, 2, and writing β for α.

Solving equations (15) for θ and w_z, we have

$$\left.\begin{aligned}(q_2-q_1)\,\theta &= q_2\left(\frac{d\Phi_1}{dx}+\frac{d\Psi_1}{dy}+\frac{dX_1}{dz}\right)-q_1\left(\frac{d\Phi_2}{dx}+\frac{d\Psi_2}{dy}+\frac{dX_2}{dz}\right)\\ (q_2-q_1)\,w_z &= -\left(\frac{d\Phi_1}{dx}+\frac{d\Psi_1}{dy}+\frac{dX_1}{dz}\right)+\left(\frac{d\Phi_2}{dx}+\frac{d\Psi_2}{dy}+\frac{dX_2}{dz}\right)\\ R &= F\theta+(C-F)\,w_z\\ &= \alpha\left(\frac{d\Phi_1}{dx}+\frac{d\Psi_1}{dy}+\frac{dX_1}{dz}\right)+\beta\left(\frac{d\Phi_2}{dx}+\frac{d\Psi_2}{dy}+\frac{dX_2}{dz}\right)\end{aligned}\right\}. \quad (16)$$

The value of ϖ is found in a similar manner. We have

$$\frac{d^2\varpi}{dx^2}+\frac{d^2\varpi}{dy^2}+\gamma_3\frac{d^2\varpi}{dz^2}=0,$$

where $$\gamma_3 = L/N,$$

and the boundary-condition is

$$2L\varpi_z = S'_{x'}-T'_{y'}.$$

Hence $$4\pi\varpi = -\frac{1}{\sqrt{LN}}\iint (S'_{x'}-T'_{y'})\,\frac{1}{r_3}\,dx'dy'$$

where $$r_3^2 = (x-x')^2+(y-y')^2+z^2/\gamma_3.$$

We may put the equation into the form

$$2\varpi = \frac{d\Psi_3}{dx}-\frac{d\Phi_3}{dy}, \quad (17)$$

where $$\Phi_3 = -\frac{1}{2\pi\sqrt{LN}}\iint T'\,\frac{1}{r_3}\,dx'dy',$$

$$\Psi_3 = -\frac{1}{2\pi\sqrt{LN}}\iint S'\,\frac{1}{r_3}\,dx'dy'.$$

To determine u, v, we have the equations

$$\begin{aligned}(q_2-q_1)(u_x+v_y) &= (q_2-q_1)(\theta-w_z)\\ &= (1+q_2)\,V_1-(1+q_1)\,V_2,\end{aligned}$$

$$v_x-u_y = \frac{d\Psi_3}{dx}-\frac{d\Phi_3}{dy}.$$

Therefore

$$(q_2-q_1)\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right)$$

$$= (1+q_2)\frac{dV_1}{dx}-(1+q_1)\frac{dV_2}{dx}-(q_2-q_1)\frac{d^2\Psi_2}{dx\,dy}+(q_2-q_1)\frac{d^2\Phi_2}{dy^2}.$$

Now put

$$\Phi_1^{ii}=\frac{1}{2\pi a}\,\frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint T'\left\{\frac{z}{\sqrt{\gamma_1}}\log\left(\frac{z}{\sqrt{\gamma_1}}+r_1\right)-r_1\right\}dx'dy',$$

so that

$$\Phi_1=\gamma_1\frac{d^2\Phi_1^{ii}}{dz^2}$$

$$=-\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\Phi_1.$$

Making the corresponding changes in the other functions and removing the operator $d^2/dx^2+d^2/dy^2$, we see that

$$(q_2-q_1)\,u=-(1+q_2)\frac{d}{dx}\left(\frac{d\Phi_1^{ii}}{dx}+\frac{d\Psi_1^{ii}}{dy}+\frac{dX_1^{ii}}{dz}\right)$$

$$+(1+q_1)\frac{d}{dx}\left(\frac{d\Phi_2^{i}}{dx}+\frac{d\Psi_2^{ii}}{dy}+\frac{dX_2^{ii}}{dz}\right)$$

$$+(q_2-q_1)\frac{d}{dy}\left(\frac{d\Psi_2^{ii}}{dx}-\frac{d\Phi_2^{ii}}{dy}\right), \qquad (18)$$

for this is a solution of the equation and it makes u vanish at infinity. Similarly,

$$(q_2-q_1)\,v=-(1+q_2)\frac{d}{dy}\left(\frac{d\Phi_1^{ii}}{dx}+\frac{d\Psi_1^{ii}}{dy}+\frac{dX_1^{ii}}{dz}\right)$$

$$+(1+q_1)\frac{d}{dy}\left(\frac{d\Phi_2^{ii}}{dx}+\frac{d\Psi_2^{ii}}{dy}+\frac{dX_2^{ii}}{dz}\right)$$

$$-(q_2-q_1)\frac{d}{dx}\left(\frac{d\Psi_2^{ii}}{dx}-\frac{d\Phi_2^{ii}}{dy}\right). \qquad (19)$$

Surface-Depression due to Normal Pressure.

5. If, in the second of equations (16), we put

$$T'=S'=0;$$

and therefore also $\Phi_1=\Psi_1=\Phi_2=\Psi_2=0$, we have

$$(q_2-q_1)\,w = -X_1+X_2$$

$$= -\frac{1}{2\pi\alpha}\,\frac{\gamma_1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint R'\,\frac{1}{r_1}\,dx'dy'$$

$$-\frac{1}{2\pi\beta}\,\frac{\gamma_2}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\iint R'\,\frac{1}{r_2}\,dx'dy'.$$

On the surface this takes the form

$$2\pi\,(q_2-q_1)\,w = -\frac{1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\left(\frac{\gamma_1}{\alpha}+\frac{\gamma_2}{\beta}\right)\iint R'\,\frac{1}{r}\,dx'dy';$$

or, substituting the values of the constants in terms of the elastic moduli,

$$w = -\frac{\sqrt{A}}{2\pi\sqrt{L}}\,\frac{\{(\sqrt{AC}+L)^2-(F+L)^2\}^{\frac{1}{2}}}{AC-F^2}\iint R'\,\frac{1}{r}\,dx'dy'.$$

Thus, for a concentrated load W, we have

$$w = \frac{\sqrt{A}}{2\pi\sqrt{L}}\,\frac{\{(\sqrt{AC}+L)^2-(F+L)^2\}^{\frac{1}{2}}}{AC-F^2}\,\frac{W}{r}.$$

For an isotropic solid

$$F = \lambda,$$

$$L = \mu,$$

$$A = C = \lambda+2\mu,$$

and hence

$$w = \frac{1}{4\pi}\,\frac{\lambda+2\mu}{\mu\,(\lambda+\mu)}\,\frac{W}{r},$$

which is Boussinesq's result.

It appears, therefore, that the law of depression is the same as for an isotropic solid; consequently, the applications of this law, which were made by Boussinesq and Hertz to problems concerning isotropic bodies in contact, may be at once extended to the æolotropic solids here considered, with the limitation that the normal to the plane of contact must be an axis of elastic symmetry.

Stress across Planes Parallel to the Boundary.

6. It is interesting to examine how the pressure due to a concentrated load W is transmitted across a plane z. From the symmetry it will be sufficient to consider the stresses R, T at $(x, 0, z)$, the load being applied at the origin.

Now
$$(q_2-q_1)\,u_x = -\frac{1+q_2}{\gamma_1}\frac{dX_1}{dx}+\frac{1+q_1}{\gamma_2}\frac{dX_2}{dx}$$

and
$$(q_2-q_1)\,w_x = -\frac{dX_1}{dx}+\frac{dX_2}{dx}.$$

Hence
$$(q_2-q_1)\,T = -L\frac{1+q_2+\gamma_1}{\gamma_1}\frac{dX_1}{dx}+L\frac{1+q_1+\gamma_2}{\gamma_2}\frac{dX_2}{dx}.$$

Also
$$R = \alpha\frac{dX_1}{dz}+\beta\frac{dX_2}{dz}.$$

Now it is readily shown that

$$\frac{1+q_2+\gamma_1}{(q_2-q_1)\,\alpha} = -\frac{1+q_1+\gamma_2}{(q_2-q_1)\,\beta} = -\frac{1}{L}.$$

Hence
$$T = \frac{\alpha}{\gamma_1}\frac{dX_1}{dx}+\frac{\beta}{\gamma_2}\frac{dX_2}{dx},$$

where
$$X_1 = -\frac{1}{2\pi\alpha}\,\frac{\gamma_1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,\frac{W}{r_1},$$

$$X_2 = \frac{1}{2\pi\beta}\,\frac{\gamma_2}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,\frac{W}{r_2};$$

and therefore
$$\frac{dX_1}{dx} = \frac{1}{2\pi\alpha}\,\frac{\gamma_1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,W\frac{x}{r_1^3},$$

$$\frac{dX_1}{dz} = \frac{1}{2\pi\alpha}\cdot\frac{1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}}\,W\frac{z}{r_1^3},$$

... ...

Thus
$$T = \frac{Wx}{2\pi\,(\sqrt{\gamma_2}-\sqrt{\gamma_1})}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)$$

and
$$R = \frac{Wz}{2\pi\,(\sqrt{\gamma_2}-\sqrt{\gamma_1})}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right),$$

where
$$\sqrt{\gamma_2}-\sqrt{\gamma_1} = (\gamma_1+\gamma_2-2\sqrt{\gamma_1\gamma_2})^{\frac{1}{2}}$$
$$= \left(\frac{AC-F^2-2FL}{AL}-2\sqrt{\frac{C}{A}}\right)^{\frac{1}{2}}$$
$$= \frac{\{(\sqrt{AC}-L)^2-(F+L)^2\}^{\frac{1}{2}}}{\sqrt{AL}}.$$

Since, now,
$$T/R = x/z,$$

it follows that the stress across the plane z is at each point directed radially from the point of application of the load. The magnitude of this stress is

$$\frac{W}{2\pi}\frac{\sqrt{AL}\,r}{\{(\sqrt{AC}-L)^2-(F+L)^2\}^{\frac{1}{2}}}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right),$$

where

$$r^2 = x^2+y^2+z^2,$$

$$r_1^2 = x^2+y^2+z^2/\gamma_1,$$

$$r_2^2 = x^2+y^2+z^2/\gamma_2.$$

To pass to the case of the isotropic solid, put

$$\gamma_1 = 1+\epsilon_1, \quad \gamma_2 = 1+\epsilon_2,$$

where ϵ_1, ϵ_2 ultimately vanish. Then

$$\frac{1}{r_1^3}-\frac{1}{r_2^3} = \frac{3z^2}{2r^5}(\epsilon_1-\epsilon_2)+\&c.$$

and

$$\frac{1}{\sqrt{\gamma_2}-\sqrt{\gamma_1}} = \frac{2}{\epsilon_2-\epsilon_1}+\&c.$$

Hence in the limit the stress becomes

$$\frac{3W}{2\pi}\frac{z^2}{r^4},$$

and this is Boussinesq's result.

Some Quadrature-Formulæ. *By* W. F. Sheppard, M.A., LL.M.

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I. *Equidistant Bounding Ordinates.*

1. The formulæ to be considered are those giving the area of a curve in terms of a series of equidistant ordinates, the two extreme ordinates forming the boundary of the area in question. Thus, if

$$z_0,\ z_1,\ z_2,\ \ldots,\ z_m$$

denote the ordinates, at successive distances h, and if the corresponding abscissæ are

$$x_0,\ x_1 = x_0+h,\ x_2 = x_0+2h,\ \ldots,\ x_m = x_0+mh,$$

the area to be determined is

$$A = \int_{x_0}^{x_m} f(x)\,dx,$$

where

$$\left.\begin{aligned} z_0 &= f(x_0) \\ z_1 &= f(x_1) \\ \vdots\ \ &\quad \vdots \\ z_m &= f(x_m) \end{aligned}\right\}.$$

It is assumed that $z \equiv f(x)$ is positive throughout the whole range considered.

The most simple rule is the *trapezoidal rule*, which gives

$$A = h\left(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m\right). \tag{1}$$

Simpson's rule, for cases in which m is even, gives

$$A = \tfrac{1}{3}h\,(z_0+4z_1+2z_2+4z_3+2z_4+\ldots+2z_{m-2}+4z_{m-1}+z_m), \tag{2}$$

the coefficients from z_1 to z_{m-2} being alternately 4 and 2.

Simpson's second rule, for cases in which m is a multiple of 3, gives

$$A = \tfrac{3}{8}h\,(z_0+3z_1+3z_2+2z_3+3z_4+\ldots+2z_{m-3}+3z_{m-2}+3z_{m-1}+z_m), \tag{3}$$

the sequence of coefficients from z_1 to z_{m-3} being 3, 3, 2.

Weddle's rule, for cases in which m is a multiple of 6, gives

$$\begin{aligned} A = \tfrac{3}{10}h\,(z_0&+5z_1+z_2+6z_3+z_4+5z_5+2z_6+\ldots \\ &\ldots+2z_{m-6}+5z_{m-5}+z_{m-4}+6z_{m-3}+z_{m-2}+5z_{m-1}+z_m), \end{aligned} \tag{4}$$

the sequence of coefficients from z_1 to z_{m-6} being 5, 1, 6, 1, 5, 2.

These are the rules that are best known.

2. The latter three rules are usually regarded as being obtained as follows.

For Simpson's rule we pass a parabola through the tops of the ordinates z_0, z_1, z_2; and we find that the area bounded by this parabola and the ordinates z_0 and z_2 is

$$\tfrac{1}{3}h\,(z_0+4z_1+z_2).$$

S 2

This area is substituted for the true area of this portion of the curve. Similarly, for the area between z_2 and z_4, we get

$$\tfrac{1}{3}h\,(z_2+4z_3+z_4),$$

and so on. Adding together all these areas, we get (2).

In the same way, for Simpson's second rule, we pass a parabola of the third degree through the tops of four consecutive ordinates, and obtain, for the area bounded by z_0 and z_3,

$$\tfrac{3}{8}h\,(z_0+3z_1+3z_2+z_3).$$

Proceeding as before, we arrive at (3).

Weddle's rule is obtained in the same way, by passing a parabola of the sixth degree through the tops of seven consecutive ordinates; but the expression for the area of this parabola is slightly altered, so as to give a result which is simpler for purposes of numerical computation.

The formulæ may, however, be regarded in another way. Let A_1 denote the area given by the trapezoidal rule (1), so that

$$A_1 = h\,(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m). \tag{5}$$

Now suppose that m is even; and let A_2 denote what A_1 would become if we left out of consideration the alternate ordinates $z_1, z_3, \ldots, z_{m-1}$. Then we have

$$A_2 = 2h\,(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-2}+\tfrac{1}{2}z_m), \tag{6}$$

and Simpson's rule may be written

$$A = \frac{4A_1-A_2}{3}. \tag{7}$$

Again, suppose that m is a multiple of 3; and let A_3 denote what A_1 would become if we left out of consideration the ordinates $z_1, z_2, z_4, z_5, \ldots, z_{m-4}, z_{m-2}, z_{m-1}$, taking only every third ordinate. Then

$$A_3 = 3h\,(\tfrac{1}{2}z_0+z_3+z_6+\ldots+z_{m-3}+\tfrac{1}{2}z_m), \tag{8}$$

and Simpson's second rule may be written

$$A = \frac{9A_1-A_3}{8}. \tag{9}$$

Finally, let m be a multiple of 6. Then Weddle's rule may be written

$$A = \frac{15A_1-6A_2+A_3}{10}. \tag{10}$$

3. Let z', z'', ... denote the differential coefficients of $z \equiv f(x)$ with regard to x. Then the *exact* area under consideration is given by the Euler-Maclaurin formula

$$A = h\left(\tfrac{1}{2}z_0 + z_1 + z_2 + \ldots + z_{m-1} + \tfrac{1}{2}z_m\right)$$
$$+\left[-\frac{B_1}{2!}h^2z' + \frac{B_2}{4!}h^4z''' - \frac{B_3}{6!}h^6z^{\mathrm{v}} + \ldots\right]_{x=x_0}^{x=x_m}, \quad (11)$$

where B_1, B_2, B_3, ... are Bernoulli's numbers, and

$$\left[\phi(x)\right]_{x=x_0}^{x=x_m}$$

denotes $\phi(x_m) - \phi(x_0)$.

Thus, with our previous notation, we have

$$A = A_1 + \left[-\frac{B_1}{2!}h^2z' + \frac{B_2}{4!}h^4z''' - \frac{B_3}{6!}h^6z^{\mathrm{v}} + \ldots\right]_{x=x_0}^{x=x_m}. \quad (12)$$

The formula (12) is exact; and we shall also get an exact expression by the same formula in terms of z_0, z_2, ..., z_m, when m is even. Thus, writing $2h$ for h, we have

$$A = A_2 + \left[-\frac{B_1}{2!}2^2h^2z' + \frac{B_2}{4!}2^4h^4z''' - \frac{B_3}{6!}2^6h^6z^{\mathrm{v}} + \ldots\right]_{x=x_0}^{x=x_m}. \quad (13)$$

Multiplying (12) and (13) by $\frac{4}{3}$ and $-\frac{1}{3}$ respectively, and adding, we find

$$A = \frac{4A_1 - A_2}{3} + \frac{1}{3}\left[-\frac{B_2}{4!}(2^4 - 2^2)h^4z''' + \frac{B_3}{6!}(2^6 - 2^2)h^6z^{\mathrm{v}} - \ldots\right]_{x=x_0}^{x=x_m}. \quad (14)$$

Comparing this with (7), we see the amount of error involved in using Simpson's rule (2). Similarly, for Simpson's second rule, we have as the true formula

$$A = \frac{9A_1 - A_3}{8} + \frac{1}{8}\left[-\frac{B_2}{4!}(3^4 - 3^2)h^4z''' + \frac{B_3}{6!}(3^6 - 3^2)h^6z^{\mathrm{v}} - \ldots\right]_{x=x_0}^{x=x_m}, \quad (15)$$

and for Weddle's rule

$$A = \frac{15A_1 - 6A_2 + A_3}{10} + \frac{1}{10}\left[-\frac{B_3}{6!}(3^6 - 6.2^6 + 15.1^6)h^6z^{\mathrm{v}} + \ldots\right]_{x=x_0}^{x=x_m}. \quad (16)$$

4. To extend these formulæ, let f denote any one of the factors of m (including 1 and m), and let us write

$$A_f = fh\left(\tfrac{1}{2}z_0 + z_f + z_{2f} + \dots + z_{m-f} + \tfrac{1}{2}z_m\right). \tag{17}$$

Then, if $a, b, c, \dots$ are different values of f, and if, for convenience of printing, we write

$$\lambda_r = B_r/(2r)! \tag{18}$$

we have
$$\left.\begin{aligned} A &= A_a + \Big[-\lambda_1 a^2h^2z' + \lambda_2 a^4h^4z''' - \lambda_3 a^6h^6z^{\mathrm{v}} + \dots\Big]_{x=x_0}^{x=x_m} \\ A &= A_b + \Big[-\lambda_1 b^2h^2z' + \lambda_2 b^4h^4z''' - \lambda_3 b^6h^6z^{\mathrm{v}} + \dots\Big]_{x=x_0}^{x=x_m} \\ A &= A_c + \Big[-\lambda_1 c^2h^2z' + \lambda_2 c^4h^4z''' - \lambda_3 c^6h^6z^{\mathrm{v}} + \dots\Big]_{x=x_0}^{x=x_m} \\ \vdots \quad & \quad \vdots \end{aligned}\right\}. \tag{19}$$

Multiplying these equations by $p, q, r, \dots$, and adding, we get

$$\begin{aligned} (p+q+r+\dots)A &= pA_a + qA_b + rA_c + \dots \\ &\dots + \Big[-(pa^2+qb^2+rc^2+\dots)\lambda_1 h^2z' \\ &\qquad + (pa^4+qb^4+rc^4+\dots)\lambda_2 h^4z''' - \dots\Big]_{x=x_0}^{x=x_m}. \end{aligned} \tag{20}$$

Hence, if there are i factors $a, b, c, \dots$, and if we take $p, q, r, \dots$, to satisfy the equations

$$\left.\begin{aligned} pa^2 + qb^2 + rc^2 + \dots &= 0 \\ pa^4 + qb^4 + rc^4 + \dots &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ pa^{2i-2} + qb^{2i-2} + rc^{2i-2} + \dots &= 0 \end{aligned}\right\}, \tag{21}$$

we have

$$\begin{aligned} (p+q+r+\dots)A &= pA_a + qA_b + rA_c + \dots \\ &+ \Big[(-)_i (pa^{2i}+qb^{2i}+rc^{2i}+\dots)\frac{B_i}{(2i)!}h^{2i}\frac{d^{2i-1}z}{dx^{2i-1}} \\ &+ (-)^{i+1}(pa^{2i+2}+qb^{2i+2}+rc^{2i+2}+\dots)\frac{B_{i+1}}{(2i+2)!}h^{2i+2}\frac{d^{2i+1}z}{dx^{2i+1}} + \dots\Big]_{x=x_0}^{x=x_m}. \end{aligned} \tag{22}$$

Thus, by using
$$A = \frac{pA_a + qA_b + rA_c + \dots}{p+q+r+\dots} \tag{23}$$
as an approximate formula, we introduce an error which only involves *differential coefficients* of z from the $(2i-1)$th onwards. Or, if we

regard the differential coefficients as expressed in terms of differences, the error introduced depends only on the $(2i-1)$th and higher differences.

The equations (21) give

$$\frac{1}{p}:\frac{1}{q}:\frac{1}{r}:\dots :: a^2(a^2-b^2)(a^2-c^2)\dots : b^2(b^2-a^2)(b^2-c^2)\dots$$
$$\dots : c^2(c^2-a^2)(c^2-b^2)\dots : \dots ; \quad (24)$$

and, in particular, for $i=2$,

$$p:q :: b^2 : -a^2. \quad (25)$$

Substituting from (24) in (23) and (22), we get the approximate formula and the error in the approximation.

5. The following are some special cases, a being taken equal to 1 throughout.

(i.) $m = M(2)$; $a=1$, $b=2$. *Simpson's rule.*—

$$p:q :: 4 : -1,$$
$$A = \tfrac{1}{3}(4A_1-A_2) = A_1+\tfrac{1}{3}(A_1-A_2).$$

(ii.) $m = M(3)$; $a=1$, $b=3$. *Simpson's second rule.*—

$$p:q :: 9 : -1,$$
$$A = \tfrac{1}{8}(9A_1-A_3) = A_1+\tfrac{1}{8}(A_1-A_3).$$

(iii.) $m = M(4)$; $a=1$, $b=2$, $c=4$.—

$$p:q:r :: 64 : -20 : 1,$$
$$A = \tfrac{1}{45}(64A_1-20A_2+A_4) = A_1+\tfrac{4}{9}(A_1-A_2)-\tfrac{1}{45}(A_1-A_4).$$

This is equivalent to the continued repetition of the formula for the area of a curve in terms of five equidistant ordinates [Boole and Moulton, *Finite Differences*, p. 47, formula (21)].

(iv.) $m = M(5)$; $a=1$, $b=5$.—

$$p:q :: 25 : -1,$$
$$A = \tfrac{1}{24}(25A_1-A_5) = A_1+\tfrac{1}{24}(A_1-A_5).$$

(v.) $m = M(6)$; $a=1$, $b=2$, $c=3$.—

$$p:q:r :: 15 : -6 : 1,$$
$$A = \tfrac{1}{10}(15A_1-6A_2+A_3) = A_1+\tfrac{1}{2}(A_1-A_2)-\tfrac{1}{10}(A_2-A_3).$$

This is Weddle's rule.

(vi.) $m = M(6)$; $a = 1$, $b = 2$, $c = 3$, $d = 6$.—

$$p : q : r : s :: 1296 : -567 : 112 : -1,$$

$$A = \tfrac{1}{840}(1296A_1 - 567A_2 + 112A_3 - A_6)$$
$$= A_1 + \tfrac{19}{35}(A_1 - A_2) - \tfrac{2}{15}(A_2 - A_3) + \tfrac{1}{840}(A_2 - A_6).$$

(vii.) $m = M(8)$; $a = 1$, $b = 2$, $c = 4$, $d = 8$.—

$$p : q : r : s :: 4096 : -1344 : 84 : -1,$$

$$A = \tfrac{1}{2835}(4096A_1 - 1344A_2 + 84A_4 - A_8)$$
$$= A_1 + \tfrac{28}{27}\left\{\tfrac{3}{7}(A_1 - A_2) - \tfrac{1}{35}(A_2 - A_4)\right\} + \tfrac{1}{2835}(A_1 - A_8).$$

(viii.) $m = M(9)$; $a = 1$, $b = 3$, $c = 9$.—

$$p : q : r :: 729 : -90 : 1,$$

$$A = \tfrac{1}{640}(729A_1 - 90A_3 + A_9) = A_1 + (1 + \tfrac{1}{8})\tfrac{1}{8}(A_1 - A_3) - \tfrac{1}{640}(A_1 - A_9).$$

(ix.) $m = M(10)$; $a = 1$, $b = 2$, $c = 5$.—

$$p : q : r :: 175 : -50 : 1,$$

$$A = \tfrac{1}{126}(175A_1 - 50A_2 + A_5) = A_1 + (1 + \tfrac{1}{6})\tfrac{1}{3}(A_1 - A_2) - \tfrac{1}{126}(A_2 - A_5).$$

(x.) $m = M(10)$; $a = 1$, $b = 2$, $c = 5$, $d = 10$.—

$$p : q : r : s :: 70000 : -20625 : 528 : -7,$$

$$A = \tfrac{1}{49896}(70000A_1 - 20625A_2 + 528A_5 - 7A_{10})$$
$$= A_1 + (\tfrac{1}{2} - \tfrac{1}{12} - \tfrac{1}{72})(A_1 - A_2) - \tfrac{2}{189}(A_2 - A_5) + \tfrac{1}{7128}(A_1 - A_{10}).$$

(xi.) $m = M(12)$; $a = 1$, $b = 2$, $c = 3$, $d = 4$.—

$$p : q : r : s :: 56 : -28 : 8 : -1,$$

$$A = \tfrac{1}{35}(56A_1 - 28A_2 + 8A_3 - A_4)$$
$$= A_1 + \tfrac{3}{5}(A_1 - A_2) - \tfrac{1}{5}(A_2 - A_3) + \tfrac{1}{35}(A_3 - A_4).$$

This is very convenient for purposes of calculation.

(xii.) $m = M(12)$; $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 6$.—

$$p : q : r : s : t :: 1728 : -945 : 320 : -54 : 1,$$

$$A = \tfrac{1}{1050}(1728A_1 - 945A_2 + 320A_3 - 54A_4 + A_6).$$

(xiii.) $m = M(15)$; $a = 1$, $b = 3$, $c = 5$.—

$$p : q : r :: 150 : -25 : 3,$$

$$A = \tfrac{1}{128}(150A_1 - 25A_3 + 3A_5).$$

6. To illustrate some of these formulæ, let us take

$$m = 12,$$

and
$$A = \int_0^1 \frac{dx}{1+x}.$$

The true value of this integral, to ten places of decimals, is

$$A = \cdot 69314\ 71806.$$

The thirteen ordinates are

x	z	x	z
0	1·00000 00000	7/12	·63157 89474
1/12	·92307 69231	8/12	·60000 00000
2/12	·85714 28571	9/12	·57142 85714
3/12	·80000 00000	10/12	·54545 45455
4/12	·75000 00000	11/12	·52173 91304
5/12	·70588 23529	12/12	·50000 00000
6/12	·66666 66667		

These give

$$A_1 = \cdot 69358\ 08329,$$
$$A_2 = \cdot 69487\ 73449,$$
$$A_3 = \cdot 69702\ 38095,$$
$$A_4 = \cdot 70000\ 00000,$$
$$A_6 = \cdot 70833\ 33333;$$

so that we have

Formula.	Value of A.	Error × 10^{10}.
(i.) (Simpson's rule)	·69314 86622	+14816
(ii.) (Simpson's second rule)	·69315 04608	+32802
(iii.)	·69314 72535	+ 729
(v.) (Weddle's rule)	·69314 72234	+ 428
(xi.)	·69314 71846	+ 40
(xii.)	·69314 71816	+ 10

7. In using rules such as Simpson's, we get rid of the terms in z', but at the expense of increasing subsequent terms; thus, if the factors are 1 and b, the coefficient of h^4z''' is altered from λ_2 to $-b^2\lambda_2$. Similarly, with Weddle's rule, we get rid of the terms in z' and z''', but, if the factors are 1, b, and c, the coefficient of h^6z^{v} is altered from $-\lambda_3$ to $-b^2c^2\lambda_3$.

If, therefore, we know—as in certain cases we do know—that the bounding values of z', and perhaps also those of z''' and z^{v}, are zero, the best value to take is A_1. The use of Simpson's rule will usually lead to a worse result; and the effect of Weddle's rule may be still worse. It may happen, however, that the bounding values of z' are known to be zero, and that nothing is known about those of z''' and z^{v}. We should then modify our formulæ by taking $p, q, r, \ldots$ to satisfy the equations

$$\left.\begin{array}{l} pa^4+qb^4+rc^4+\ldots=0 \\ pa^6+qb^6+rc^6+\ldots=0 \\ \quad\vdots \qquad\qquad\qquad \vdots \\ pa^{2i}+qb^{2i}+rc^{2i}+\ldots=0 \end{array}\right\},$$

which would give

$$\frac{1}{p}:\frac{1}{q}:\frac{1}{r}:\ldots::a^4(a^2-b^2)(a^2-c^2)\ldots:b^4(b^2-a^2)(b^2-c^2)\ldots$$
$$\ldots:c^4(c^2-a^2)(c^2-b^2)\ldots:\ldots;$$

and so on.

II. *Equidistant Mid-Ordinates.*

8. In some cases the known ordinates are not the bounding ordinates $z_0, z_1, z_2, \ldots, z_m$, but the intermediate ordinates $z_{\frac{1}{2}}, z_{\frac{3}{2}}, \ldots, z_{m-\frac{1}{2}}$. To express the area in terms of the latter, we have, by (11),

$$A=h\left(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m\right)+\left[-\frac{B_1}{2!}h^2z'+\frac{B_2}{4!}h^4z'''-\ldots\right]_{x=x_0}^{x=x_m},$$

and, taking intervals $\frac{1}{2}h$,

$$A=\tfrac{1}{2}h\left(\tfrac{1}{2}z_0+z_{\frac{1}{2}}+z_1+z_{\frac{3}{2}}+\ldots+z_{m-\frac{1}{2}}+\tfrac{1}{2}z_m\right)$$
$$+\left[-\frac{B_1}{2^2.2!}h^2z'+\frac{B_2}{2^4.4!}h^4z'''-\ldots\right]_{x=x_0}^{x=x_m}.$$

Multiplying the second of these formulæ by 2, and subtracting the first, we get

$$A = S_1 + \Big[(1-1/2)\frac{B_1}{2!}h^2z' - (1-1/2^3)\frac{B_2}{4!}h^4z'''$$

$$+(1-1/2^5)\frac{B_3}{6!}h^6z^{\mathrm{v}} - \dots\Big]_{x=x_0}^{x=x_m}, \quad (26)$$

where $$S_1 = h\,(z_{\frac12}+z_{\frac32}+\dots+z_{m-\frac12}). \quad (27)$$

The leading term in square brackets in (26) is half the corresponding term in (12), and therefore, if these terms are not eliminated, the use of mid-ordinates will generally give a better result than the use of bounding ordinates. But the mid-ordinates not convenient for more accurate formulæ. The reasoning of § 4 applies, but it is necessary that the factors $a, b, c, \dots$ should be odd numbers. Thus, if m is a multiple of 3, Simpson's second rule applies; so that

$$A = S_1 + \tfrac18\,(S_1 - S_3)$$

gives a better result than $A = S_1$, where

$$S_3 = 3h\,(z_{\frac32}+z_{\frac92}+\dots+z_{m-\frac32}).$$

III. *Miscellaneous Formulæ.*

9. Various methods have been devised for getting rid of the terms in z' in the Euler-Maclaurin formula, or in the corresponding formula for mid-ordinates, so as to retain the trapezoidal rule as the main basis of calculation. Thus, in Parmentier's rule, the ordinates taken are $z_{\frac12}, z_{\frac32}, z_{\frac52}, \dots, z_{m-\frac12}$, and also z_0 and z_m. The values of $\frac12 hz'$ are then taken as equal to $z_{\frac12}-z_0$ and $z_m - z_{m-\frac12}$ respectively, so that the formula becomes

$$A = S_1 + \tfrac{1}{12}h\,(z_0 - z_{\frac12} - z_{m-\frac12} + z_m). \quad (28)$$

This, however, is not a very good rule; for, although it gets rid of the terms in z', it introduces terms in z''. These latter terms can be kept out by a slight alteration in the formula.

10. Let

$$z_{\frac12 m}$$

denote the mid-ordinate of the whole area, the magnitude of this ordinate not being necessarily known. Then, if we take two ordinates

$$z_{\frac12 m(1-a)}, \quad z_{\frac12 m(1+a)}$$

at equal distances from $z_{\frac{1}{2}m}$, and if we write

$$z=f(x)$$

and

$$\tfrac{1}{2}mhd/dx=D,$$

we have

$$\begin{aligned}
z_{\frac{1}{2}m(1-a)}&+z_{\frac{1}{2}m(1+a)}\\
&=f(x_{\frac{1}{2}m}-\tfrac{1}{2}mah)+f(x_{\frac{1}{2}m}+\tfrac{1}{2}mah)\\
&=2\cosh aD.f(x_{\frac{1}{2}m})\\
&=2f(x_{\frac{1}{2}m})+\left[\frac{\cosh aD-1}{\sinh D}f(x)\right]_{x=x_0}^{x=x_m}\\
&=2z_{\frac{1}{2}m}+\Big[\frac{a^2}{2!}\tfrac{1}{2}mhz'-\frac{2a^2-a^4}{4!}(\tfrac{1}{2}mh)^3z'''+\frac{7a^2-5a^4+a^6}{6!}(\tfrac{1}{2}mh)^5z^{\mathrm{v}}\\
&\qquad-\frac{124a^2-98a^4+28a^6-3a^8}{3.8!}(\tfrac{1}{2}mh)^7z^{\mathrm{vii}}+\dots\Big]_{x=x_0}^{x=x_m}. \qquad (29)
\end{aligned}$$

Taking another pair of ordinates

$$z_{\frac{1}{2}m(1-\beta)},\quad z_{\frac{1}{2}m(1+\beta)},$$

we get a similar formula. Eliminating $z_{\frac{1}{2}m}$ between the two formulæ, we find

$$\begin{aligned}
&z_{\frac{1}{2}m(1-a)}-z_{\frac{1}{2}m(1-\beta)}-z_{\frac{1}{2}m(1+\beta)}+z_{\frac{1}{2}m(1+a)}\\
&=(a^2-\beta^2)\Big[\tfrac{1}{4}mhz'-\frac{2-(a^2+\beta^2)}{192}m^3h^3z'''+\frac{7-5(a^2+\beta^2)+(a^4+a^2\beta^2+\beta^4)}{23040}m^5h^5z^{\mathrm{v}}\\
&\qquad-\frac{124-98(a^2+\beta^2)+28(a^4+a^2\beta^2+\beta^4)-3(a^2+\beta^2)(a^4+\beta^4)}{15482880}m^7h^7z^{\mathrm{vii}}+\dots\Big]_{x=x_0}^{x=x_m}.
\end{aligned}$$

(30)

This formula can be used for eliminating the terms in z', z''', ... from (11) or (26). Thus, if we take two pairs of ordinates, we should get as our approximate formulæ

$$\begin{aligned}
A&=h(\tfrac{1}{2}z_0+z_1+z_2+\dots+z_{m-1}+\tfrac{1}{2}z_m)\\
&\qquad-\frac{1}{3m(a^2-\beta^2)}h(z_{\frac{1}{2}m(1-a)}-z_{\frac{1}{2}m(1-\beta)}-z_{\frac{1}{2}m(1+\beta)}+z_{\frac{1}{2}m(1+a)}), \qquad (31)\\
A&=h(z_{\frac{1}{2}}+z_{\frac{3}{2}}+z_{\frac{5}{2}}+\dots+z_{m-\frac{1}{2}})\\
&\qquad+\frac{1}{6m(a^2-\beta^2)}h(z_{\frac{1}{2}m(1-a)}-z_{\frac{1}{2}m(1-\beta)}-z_{\frac{1}{2}m(1+\beta)}+z_{\frac{1}{2}m(1+a)}). \qquad (32)
\end{aligned}$$

Or, if we take three pairs of ordinates, we shall get

$$A = h\left(\tfrac{1}{2}z_0+z_1+z_2+\ldots+z_{m-1}+\tfrac{1}{2}z_m\right)$$
$$-\frac{2-4/(5m^2)-(\beta^2+\gamma^2)}{3m\,(\alpha^2-\beta^2)(\alpha^2-\gamma^2)}\,h\left(z_{\frac{1}{2}m(1-\alpha)}+z_{\frac{1}{2}m(1+\alpha)}\right)$$
$$-\frac{2-4/(5m^2)-(\gamma^2+\alpha^2)}{3m\,(\beta^2-\gamma^2)(\beta^2-\alpha^2)}\,h\left(z_{\frac{1}{2}m(1-\beta)}+z_{\frac{1}{2}m(1+\beta)}\right)$$
$$-\frac{2-4/(5m^2)-(\alpha^2+\beta^2)}{3m\,(\gamma^2-\alpha^2)(\gamma^2-\beta^2)}\,h\left(z_{\frac{1}{2}m(1-\gamma)}+z_{\frac{1}{2}m(1+\gamma)}\right), \tag{33}$$

$$A = h\left(z_{\frac{1}{2}}+z_{\frac{3}{2}}+z_{\frac{5}{2}}+\ldots+z_{m-\frac{1}{2}}\right)$$
$$+\frac{2-7/(5m^2)-(\beta^2+\gamma^2)}{6m\,(\alpha^2-\beta^2)(\alpha^2-\gamma^2)}\,h\left(z_{\frac{1}{2}m(1-\alpha)}+z_{\frac{1}{2}m(1+\alpha)}\right)$$
$$+\frac{2-7/(5m^2)-(\gamma^2+\alpha^2)}{6m\,(\beta^2-\gamma^2)(\beta^2-\alpha^2)}\,h\left(z_{\frac{1}{2}m(1-\beta)}+z_{\frac{1}{2}m(1+\beta)}\right)$$
$$+\frac{2-7/(5m^2)-(\alpha^2+\beta^2)}{6m\,(\gamma^2-\alpha^2)(\gamma^2-\beta^2)}\,h\left(z_{\frac{1}{2}m(1-\gamma)}+z_{\frac{1}{2}m(1+\gamma)}\right). \tag{34}$$

If we take only the two pairs of ordinates, α and β should be as nearly equal to 1 as possible, in order that the new term in z''' may be as small as possible. Thus, if we only know the ordinates z_0, z_1, …, z_m, we have $\alpha = 1$, $\beta = 1-2/m$, and (31) gives

$$A = A_1-\tfrac{1}{12}\frac{m}{m-1}\,h\left(z_0-z_1-z_{m-1}+z_m\right). \tag{35}$$

Similarly, if we only know the mid-ordinates, we have $\alpha = 1-1/m$, $\beta = 1-3/m$, and (32) gives

$$A = S_1+\tfrac{1}{24}\frac{m}{m-2}\,h\left(z_{\frac{1}{2}}-z_{\frac{3}{2}}-z_{m-\frac{3}{2}}+z_{m-\frac{1}{2}}\right). \tag{36}$$

In the same way, if we know the bounding ordinates z_0 and z_{m-1} and the extreme mid-ordinates $z_{\frac{1}{2}}$ and $z_{m-\frac{1}{2}}$, we find

$$A = A_1-\tfrac{1}{6}\,\frac{2m}{2m-1}\,h\left(z_0-z_{\frac{1}{2}}-z_{m-\frac{1}{2}}+z_m\right), \tag{37}$$

$$A = S_1+\tfrac{1}{12}\frac{2m}{2m-1}\,h\left(z_0-z_{\frac{1}{2}}-z_{m-\frac{1}{2}}+z_m\right). \tag{38}$$

This last is a simple modification of Parmentier's rule, and it will be found to give decidedly better results.

Similarly, from (33) and (34), we can get a number of formulæ, of which the following are examples :—

$$A = A_1 - \tfrac{1}{120}\frac{m(15m-26)}{(m-1)(m-2)}h(z_0-z_1-z_{m-1}+z_m)$$

$$+\tfrac{1}{120}\frac{m(5m-6)}{(m-2)(m-3)}h(z_1-z_2-z_{m-2}+z_{m-1}), \qquad (39)$$

$$A = S_1 + \tfrac{1}{960}\frac{m(80m-177)}{(m-2)(m-3)}h(z_{\frac{1}{2}}-z_{\frac{3}{2}}-z_{m-\frac{3}{2}}+z_{m-\frac{1}{2}})$$

$$-\tfrac{1}{960}\frac{m(40m-57)}{(m-3)(m-4)}h(z_{\frac{3}{2}}-z_{\frac{5}{2}}-z_{m-\frac{5}{2}}+z_{m-\frac{3}{2}}), \qquad (40)$$

$$A = A_1 - \tfrac{1}{60}\frac{m(30m-29)}{(m-1)(2m-1)}h(z_0-z_{\frac{1}{2}}-z_{m-\frac{1}{2}}-z_m)$$

$$+\tfrac{1}{60}\frac{m(10m-9)}{(m-1)(2m-3)}h(z_{\frac{1}{2}}-z_1-z_{m-1}+z_{m-\frac{1}{2}}), \qquad (41)$$

$$A = S_1 + \tfrac{1}{90}\frac{m(40m-57)}{(2m-1)(2m-3)}h(z_0-z_{\frac{1}{2}}-z_{m-\frac{1}{2}}+z_m)$$

$$-\tfrac{1}{180}\frac{m(5m-6)}{(m-2)(2m-3)}h(z_{\frac{1}{2}}-z_{\frac{3}{2}}-z_{m-\frac{3}{2}}+z_{m-\frac{1}{2}}). \qquad (42)$$

The method can, of course, be extended to any number of pairs of ordinates. Thus we should get, for three pairs,

$$A = A_1 - \frac{mh}{5040}\left\{\frac{763m^2-3444m+3636}{(m-1)(m-2)(m-3)}(z_0-z_1-z_{m-1}+z_m)\right.$$

$$-\frac{4(119m^2-504m+432)}{(m-2)(m-3)(m-4)}(z_1-z_2-z_{m-2}+z_{m-1})$$

$$\left.+\frac{133m^2-462m+360}{(m-3)(m-4)(m-5)}(z_2-z_3-z_{m-3}+z_{m-2})\right\}, \qquad (43)$$

$$A = S_1 + \frac{mh}{80640}\left\{\frac{9842m^2-53970m+70407}{(m-2)(m-3)(m-4)}(z_{\frac{1}{2}}-z_{\frac{3}{2}}-z_{m-\frac{3}{2}}+z_{m-\frac{1}{2}})\right.$$

$$-\frac{9604m^2-46032m+45810}{(m-3)(m-4)(m-5)}(z_{\frac{3}{2}}-z_{\frac{5}{2}}-z_{m-\frac{5}{2}}+z_{m-\frac{3}{2}})$$

$$\left.+\frac{3122m^2-12222m+10935}{(m-4)(m-5)(m-6)}(z_{\frac{5}{2}}-z_{\frac{7}{2}}-z_{m-\frac{7}{2}}+z_{m-\frac{5}{2}})\right\}. \qquad (44)$$

These formulæ look clumsy; but the coefficients can be calculated, *and their values* tabulated, for various values of m.

11. The formulæ obtained in the last section are quite different in character from those obtained in § 4; but both sets of formulæ, and all other quadrature-formulæ depending on symmetrically-placed ordinates (whether equidistant or not), can be comprised in one general relation.

Taking, as in § 10, two ordinates

$$z_{\frac{1}{2}m(1-a)},\quad z_{\frac{1}{2}m(1+a)},$$

we have, with the notation of that section,

$$\begin{aligned}
A &= \int_{-\frac{1}{2}mh}^{\frac{1}{2}mh} f(x_{\frac{1}{2}m}+\theta)\,d\theta \\
&= mh\frac{\sinh D}{D}f(x_{\frac{1}{2}m}) \\
&= mh\cosh aD.f(x_{\frac{1}{2}m})+\tfrac{1}{2}mh\left[\left(\frac{1}{D}-\frac{\cosh aD}{\sinh D}\right)f(x)\right]_{x=x_0}^{x=x_m} \\
&= \tfrac{1}{2}mh\left(z_{\frac{1}{2}m(1-a)}+z_{\frac{1}{2}m(1+a)}\right) \\
&\quad+\left[\frac{1-3a^2}{6}(\tfrac{1}{2}mh)^2z'-\frac{7-30a^2+15a^4}{360}(\tfrac{1}{2}mh)^4z'''\right. \\
&\qquad\left.+\frac{31-147a^2+105a^4-21a^6}{15120}(\tfrac{1}{2}mh)^6z^{\text{v}}-\dots\right]_{x=x_0}^{x=x_m}. \qquad (45)
\end{aligned}$$

Replacing a by $\beta, \gamma, \dots$, we get a series of different expressions for A. Multiplying these by $p, q, r, \dots$, adding, and dividing by $p+q+r+\dots$, we get a general expression for A in terms of a series of pairs of ordinates, symmetrically placed about $z_{\frac{1}{2}m}$, with terms involving the bounding values of $z', z''', z^{\text{v}}, \dots$; and the values of $p, q, r, \dots$ can be chosen so that the coefficients of these latter terms shall satisfy given conditions.

Thus, to get rid of the differential coefficients of z up to the $(2i-1)$th inclusive, we must take $p, q, r, \dots$ to satisfy the first i of the equations

$$\left.\begin{array}{l}
p(1-3a^2) \qquad\qquad +q(1-3\beta^2) \qquad\qquad +r(1-3\gamma^2) \qquad\qquad +\dots=0 \\
p(7-30a^2+15a^4)+q(7-30\beta^2+15\beta^4)+r(7-30\gamma^2+15\gamma^4)+\dots=0 \\
\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{array}\right\}. \qquad (46)$$

We have, therefore, as an approximate expression for the area,

$$\begin{aligned}
A = \tfrac{1}{2}mh\,\{&p\,(z_{\frac{1}{2}m(1-a)}+z_{\frac{1}{2}m(1+a)})+q\,(z_{\frac{1}{2}m(1-\beta)}+z_{\frac{1}{2}m(1+\beta)}) \\
&+r\,(z_{\frac{1}{2}m(1-\gamma)}+z_{\frac{1}{2}m(1+\gamma)})+\dots\}, \qquad (47)
\end{aligned}$$

where p, q, r, ... satisfy the first $i+1$ of the equations

$$\left.\begin{aligned} p\;\;+q\;\;+r\;\;+\ldots &= 1 \\ p\alpha^2+q\beta^2+r\gamma^2+\ldots &= \tfrac{1}{3} \\ p\alpha^4+q\beta^4+r\gamma^4+\ldots &= \tfrac{1}{5} \\ p\alpha^6+q\beta^6+r\gamma^6+\ldots &= \tfrac{1}{7} \\ \vdots\qquad\qquad & \;\;\vdots \end{aligned}\right\} . \qquad (48)$$

The formula (47), with the condition given by (48), comprises all the formulæ given in the preceding sections.

12. We have assumed, in the last section, that the values of α, β, γ, ... are known. Suppose, however, that these are arbitrary, and that there are n of them. Then there will also be n of the coefficients p, q, r, ... These $2n$ quantities can be determined so as to satisfy the first $2n$ of the equations (48); and, using them in (47), we shall get an expression which agrees with A up to the $(4n-3)$th differential coefficient of z. The first differential coefficient which will appear will be the $(4n-1)$th; and, if z is of the $(4n-1)$th degree in x, this differential coefficient will be a constant, and therefore will vanish when taken between limits. Hence, if z is of the $(4n-1)$th degree in x, or of a lower (integral) degree, we can obtain an accurate value for the area by choosing $2n$ suitable ordinates. Similarly, if z is of the $(4n+1)$th degree, or of a lower degree, we only require $2n+1$ ordinates. This is Gauss's well-known theorem.

IV. *Extension to Calculation of Volumes.*

13. The method of § 4 may be extended to the calculation of volumes. Suppose that a volume stands on a rectangular base, the sides being mh and nk, and that it is divided by $m-1$ planes at distances h in one direction and $n-1$ planes at distances k in the other. Then the ordinates which we have to take may be either the edges of the mn constituent prisms, or the mid-ordinates of the faces in one direction, or the central ordinates of the prisms. Generally they will be the edges of the prisms, and they may be denoted by

$$\begin{array}{ccccc} z_{0,0} & z_{1,0} & z_{2,0} & \cdots & z_{m,0}, \\ z_{0,1} & z_{1,1} & z_{2,1} & \cdots & z_{m,1}, \\ \vdots & \vdots & \vdots & & \vdots \\ z_{0,n} & z_{1,n} & z_{2,n} & \cdots & z_{m,n}. \end{array}$$

Let $V_{1,1}$ denote hk times the sum of these ordinates, taken with coefficients

$$\begin{array}{cccccccc} \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} \end{array}$$

Then the true volume V will differ from $V_{1,1}$ by a rather complicated expression, involving the differences of bounding values of certain differential coefficients of the ordinate z. It is not necessary to write down this expression; we need only note that it consists of

$$\begin{array}{llll} \text{(i.) terms in } h^2k\dfrac{dz}{dx}, & h^4k\dfrac{d^3z}{dx^3}, & h^6k\dfrac{d^5z}{dx^5}, & \ldots; \\ \text{(ii.) terms in } hk^2\dfrac{dz}{dy}, & hk^4\dfrac{d^3z}{dy^3}, & hk^6\dfrac{d^5z}{dy^5}, & \ldots; \\ \text{(iii.) terms in } h^2k^2\dfrac{d^2z}{dx\,dy}, & h^4k^2\dfrac{d^4z}{dx^3dy}, & h^6k^2\dfrac{d^6z}{dx^5dy}, & \ldots, \\ & h^2k^4\dfrac{d^4z}{dx\,dy^3}, & h^4k^4\dfrac{d^6z}{dx^3dy^3}, & \ldots, \\ & & h^2k^6\dfrac{d^6z}{dx\,dy^5}, & \ldots, \\ & & \&c. & \end{array}$$

Now let b and β denote any factors of m and of n respectively, and let $V_{b,1}$, $V_{1,\beta}$, $V_{b,\beta}$ denote the different values of $V_{1,1}$ obtained by altering the intervals from h to bh or from k to βk. Then we may take

$$V = \frac{pV_{1,1}+qV_{b,1}+rV_{1,\beta}+sV_{b,\beta}}{p+q+r+s} \tag{49}$$

as a more accurate value of the volume, and get rid of the terms in

$$\frac{dz}{dx},\quad \frac{dz}{dy},\quad \frac{d^2z}{dx\,dy},$$

provided p, q, r, s satisfy the equations

$$\left.\begin{array}{l} p+qb^2+r\beta+sb^2\beta = 0 \\ p+qb+r\beta^2+sb\beta^2 = 0 \\ p+qb^2+r\beta^2+sb^2\beta^2 = 0 \end{array}\right\}. \tag{50}$$

These equations give

$$p : q : r : s :: b^2\beta^2 : -\beta^2 : -b^2 : 1;$$

so that our approximate formula is

$$V = \frac{b^2\beta^2 V_{1,1} - \beta^2 V_{b,1} - b^2 V_{1,\beta} + V_{b,\beta}}{(b^2-1)(\beta^2-1)}. \tag{51}$$

If, for instance, $b = \beta = 2$, we have

$$V = \tfrac{1}{9}\left(16V_{1,1} - 4V_{2,1} - 4V_{1,2} + V_{2,2}\right), \tag{52}$$

which is Simpson's rule, applied to volumes.

14. To get more accurate formulæ, by using a larger number of factors of m and n, we might proceed in the same way, and write down the equations corresponding to (50). But it is not necessary to do this. The coefficients in (51) are the coefficients of the products of A and A' in the expression

$$\frac{b^2A_1 - A_b}{b^2-1}\,\frac{\beta^2A_1' - A_\beta'}{\beta^2-1},$$

the first factor of which is the formula giving the area of a section in one direction in terms of the ordinates at intervals h and the ordinates at intervals bh, while the second factor is the corresponding formula for a section in the other direction, the intervals being k and βk. This rule can be extended, as follows :—

The true value of V is

$$\int_{x_0}^{x_m}\int_{y_0}^{y_n} z\,dx\,dy = \frac{\sinh \frac{1}{2}mh\,d/dx}{d/dx}\,\frac{\sinh \frac{1}{2}nk\,d/dy}{d/dy}\,z_{\frac{1}{2}m,\frac{1}{2}n}. \tag{53}$$

Now let $a, b, c, \dots$ denote any factors of m. Then, if we express each side of the formula (22) as the result of an operation on the mid-ordinate $z_{\frac{1}{2}m}$, we see that

$$\begin{aligned}\frac{\sinh \frac{1}{2}mh\,d/dx}{d/dx} &= \left\{\frac{p.ah\mu_a\sigma_a + q.bh\mu_b\sigma_b + r.ch\mu_c\sigma_c + \dots}{p+q+r+\dots}\right.\\ &\quad + (-)^i\frac{pa^{2i}+qb^{2i}+rc^{2i}+\dots}{p+q+r+\dots}\,\frac{B_i}{(2i)!}\,h^{2i}\,\frac{d^{2i-1}}{dx^{2i-1}}\\ &\quad \left.+ (-)^{i+1}\frac{pa^{2i+2}+qb^{2i+2}+rc^{2i+2}+\dots}{p+q+r+\dots}\,\frac{B_{i+1}}{(2i+2)!}\,h^{2i+2}\,\frac{d^{2i+1}}{dx^{2i+1}} + \dots\right\}\delta_m,\end{aligned} \tag{54}$$

where μ, δ, σ denote the central-difference operators explained in a previous paper.* These operators are defined by the relations

$$\left.\begin{aligned} \mu_f\phi(x) &= \tfrac{1}{2}\left\{\phi\left(x+\tfrac{1}{2}fh\right)+\phi\left(x-\tfrac{1}{2}fh\right)\right\} \\ \delta_f\phi(x) &= \phi\left(x+\tfrac{1}{2}fh\right)-\phi\left(x-\tfrac{1}{2}fh\right) \\ \delta_f\sigma_f\phi(x) &= \varphi(x) \end{aligned}\right\},$$

whence, if f is a factor of m,

$$\mu_f\sigma_f(z_m-z_0) = \tfrac{1}{2}z_0+z_f+z_{2f}+\ldots+z_{m-f}+\tfrac{1}{2}z_m.$$

Similarly, if α, β, γ, ... are j factors of n, and if μ', δ', σ' denote the operations μ, δ, σ, performed with regard to y, we shall have a formula

$$\frac{\sinh\frac{1}{2}nk\,d/dy}{d/dy} = \left\{\frac{p'.\alpha k\mu'_\alpha\sigma'_\alpha+q'.\beta k\mu'_\beta\sigma'_\beta+r'.\gamma k\mu'_\gamma\sigma'_\gamma+\ldots}{p'+q'+r'+\ldots}\right.$$
$$\left.+\text{terms in } \frac{d^{2j-1}}{dy^{2j-1}},\ \frac{d^{2j+1}}{dy^{2j+1}},\ \ldots\right\}\delta'_n, \quad (55)$$

where p', q', r', ... satisfy equations similar to (21). If, now, we substitute from (54) and (55) in (53), the two sets of operators μ, σ, δ and μ', σ', δ' will combine with one another, and with powers of d/dx and d/dy, according to the ordinary laws of algebra, and we shall have, as an approximate formula,

$$V = \frac{p.ah\mu_a\sigma_a+q.bh\mu_b\sigma_b+r.ch\mu_c\sigma_c+\ldots}{p+q+r+\ldots}$$
$$\times\frac{p'.\alpha k\mu'_\alpha\sigma'_\alpha+q'.\beta k\mu'_\beta\sigma'_\beta+r'.\gamma k\mu'_\gamma\sigma'_\gamma+\ldots}{p'+q'+r'+\ldots}\delta_m\delta'_n z_{\frac{1}{2}m,\frac{1}{2}n}$$
$$\begin{aligned} = \{&pp'V_{a,\alpha}+pq'V_{a,\beta}+pr'V_{a,\gamma}+\ldots \\ &+qp'V_{b,\alpha}+qq'V_{b,\beta}+qr'V_{b,\gamma}+\ldots \\ &+rp'V_{c,\alpha}+rq'V_{c,\beta}+rr'V_{c,\gamma}+\ldots \\ &+\ldots\}/(p+q+r+\ldots)(p'+q'+r'+\ldots), \end{aligned} \quad (56)$$

the ratios $p:q:r:\ldots$ and $p':q':r':\ldots$ being given by the equations

$$\left.\begin{aligned} pa^2 &+qb^2 &+rc^2 &+\ldots = 0 \\ pa^4 &+qb^4 &+rc^4 &+\ldots = 0 \\ &\vdots & & \vdots \\ pa^{2i-2}&+qb^{2i-2}&+rc^{2i-2}&+\ldots = 0 \end{aligned}\right\} \quad (57)$$

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* *Proceedings*, Vol. xxxi., p. 449.

T 2

and

$$\left.\begin{array}{l} p'\alpha^2 \;+q'\beta^2 \;+r'\gamma^2 \;+\ldots=0 \\ p'\alpha^4 \;+q'\beta^4 \;+r'\gamma^4 \;+\ldots=0 \\ \vdots \hfill \vdots \\ p'\alpha^{2\nu-2}+q'\beta^{2\nu-2}+r'\gamma^{2\nu-2}+\ldots=0 \end{array}\right\}. \qquad (58)$$

Suppose, for instance, that m is a multiple of 4, and n a multiple of 6, and that we take (iii.) of § 5 in the one case, and (v.) (Weddle's rule) in the other. Then, developing the product

$$\frac{64A_1-20A_2+A_4}{45}\cdot\frac{15A_1'-6A_2'+A_3'}{10},$$

we find, as our approximate formula,

$$\begin{aligned} V=\tfrac{1}{450}\,(960V_{1,1}&-384V_{1,2}+64V_{1,3} \\ -300V_{2,1}&+120V_{2,2}-20V_{2,3} \\ +\;15V_{4,1}&-\;\;6V_{4,2}+\;\;V_{4,3}). \end{aligned} \qquad (59)$$

15. In the last section we have only considered the case in which the given ordinates are the edges of the constituent prisms; but the method applies equally to the other cases, subject to the limitations pointed out in § 8. Thus, if the ordinates are the central ordinates

$$\begin{array}{ccccc} z_{\frac{1}{2},\frac{1}{2}} & z_{\frac{3}{2},\frac{1}{2}} & z_{\frac{5}{2},\frac{1}{2}} & \ldots & z_{m-\frac{1}{2},\frac{1}{2}} \\ z_{\frac{1}{2},\frac{3}{2}} & z_{\frac{3}{2},\frac{3}{2}} & z_{\frac{5}{2},\frac{3}{2}} & \ldots & z_{m-\frac{1}{2},\frac{3}{2}} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{\frac{1}{2},n-\frac{1}{2}} & z_{\frac{3}{2},n-\frac{1}{2}} & z_{\frac{5}{2},n-\frac{1}{2}} & \ldots & z_{m-\frac{1}{2},n-\frac{1}{2}} \end{array}$$

and if $W_{1,1}$ denotes hk times the sum of these ordinates, we shall have a formula

$$\begin{aligned} V=\{\,&pp'W_{a,\alpha}+pq'W_{a,\beta}+pr'W_{a,\gamma}+\ldots \\ &+qp'W_{b,\alpha}+qq'W_{b,\beta}+qr'W_{b,\gamma}+\ldots \\ &+rp'W_{c,\alpha}+rq'W_{c,\beta}+rr'W_{c,\gamma}+\ldots \\ &+\ldots\}/(p+q+r+\ldots)(p'+q'+r'+\ldots), \end{aligned} \qquad (60)$$

the ratios $p:q:r:\ldots$ and $p':q':r':\ldots$ being given, as before, by *(57) and (58)*; but the factors $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ must all be

odd. Similarly, if the ordinates are

$$\begin{matrix} z_{0,\frac{1}{2}} & z_{1,\frac{1}{2}} & z_{2,\frac{1}{2}} & \cdots & z_{m,\frac{1}{2}} \\ z_{0,\frac{3}{2}} & z_{1,\frac{3}{2}} & z_{2,\frac{3}{2}} & \cdots & z_{m,\frac{3}{2}} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{0,n-\frac{1}{2}} & z_{1,n-\frac{1}{2}} & z_{2,n-\frac{1}{2}} & \cdots & z_{m,n-\frac{1}{2}} \end{matrix}$$

the factors of m may be odd or even, but the factors of n must only be odd. The coefficients by which the respective ordinates are to be multiplied in this latter case are

$$\begin{matrix} \frac{1}{2} & 1 & 1 & \ldots & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & \ldots & 1 & \frac{1}{2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{2} & 1 & 1 & \ldots & 1 & \frac{1}{2} \end{matrix}$$

Second Complément à l'Analysis Situs. *Par* H. Poincaré.

Read at request of the President, June 14th, 1900, and received June 30th, 1900.

Introduction.

J'ai publié dans le *Journal de l'Ecole Polytechnique* (Tome C, N° 1) un travail intitulé "Analysis Situs"; je me suis occupé une seconde fois du même problème dans un mémoire portant pour titre "Complément à l'Analysis Situs," et qui a été imprimé dans les *Rendiconti del Circolo Matematico di Palermo* (Tome XIII, 1899).

Cependant la question est loin d'être épuisée, et je serai sans doute forcé d'y revenir à plusieurs reprises. Pour cette fois, je me bornerai à certaines considérations qui sont de nature à simplifier, à éclaircir et à compléter les résultats précédemment acquis.

Les renvois portant simplement une indication de paragraphe ou de page se rapporteront au premier mémoire, celui du *Journal de l'Ecole Polytechnique;* les renvois où ces indications seront précédées de la lettre C se rapporteront au mémoire des *Rendiconti*.

Quant aux renvois aux paragraphes du présent mémoire, je les ferai précéder des lettres 2 c.

1. *Rappel des principales Définitions.*

Considérons une variété fermée à p dimensions. Nous supposerons que cette variété a été subdivisée de manière à former un polyèdre P à p dimensions. Les éléments de ce polyèdre s'appelleront les a_i^p; ils seront séparés les uns des autres par des variétés à $p-1$ dimensions qui s'appelleront les a_i^{p-1}; celles-ci seront séparées les unes des autres par des variétés à $p-2$ dimensions qui s'appelleront les a_i^{p-2}; et ainsi de suite jusqu'à ce qu'on arrive aux sommets du polyèdre qui s'appelleront les a_i^0.

Toutes ces variétés seront simplement connexes, c'est-à-dire homéomorphes à l'hypersphère.

Si une variété a_i^q a pour frontière complète les a_j^{q-1}, j'écrirai la congruence

$$(1) \quad a_i^q \equiv \Sigma \epsilon_{ij}^q a_j^{q-1},$$

où les ϵ sont égaux à 0, +1 ou −1 (c, § 2, p. 7).

Nous écrirons d'autre part l'homologie

$$(2) \quad \Sigma \epsilon_{ij}^q a_j^{q-1} \sim 0.$$

Nous combinerons les congruences (1) et les homologies (2) par addition, soustraction, multiplication, et quelquefois par division.

Parmi les congruences entre a_i^q et a_i^{q-1} obtenues par la combinaison des congruences (1), nous distinguerons celles qui ne contiennent que des a_i^q, et d'où les a_j^{q-1} ont disparu.

Nous désignerons quelquefois les a_i^0 sous le nom de *sommets*, les a_i^1 sous celui d'*arêtes*, les a_i^2 sous celui de *faces*, les a_i^3 sous celui de *cases*, les a_i^4 sous celui d'*hypercases*.

Au polyèdre P correspond un polyèdre réciproque P' (c, § 7), dont j'appellerai les éléments b_i^p au lieu de a_i^p, b_i^{p-1} au lieu de a_i^{p-1}, ..., et enfin b_i^0 au lieu de a_i^0.

Entre les deux polyèdres, il y a une correspondance telle que b_i^{p-q} correspond à a_i^q. Les deux polyèdres proviennent de la subdivision d'une même variété V.

Entre les éléments de P', nous avons les congruences

$$(1 \text{ bis}) \quad b_i^q = \Sigma \epsilon_{ji}^{p-q+1} b_j^{q-1}$$

analogues aux congruences (1) ; nous pouvons l'écrire également

$$b_i^q = \Sigma \epsilon_{ij}'^q b_j^{q-1},$$

en posant

$$\epsilon_{ji}^{p-q+1} = \epsilon_{ij}'^q.$$

Entre les éléments de P et ceux de P', nous avons encore une autre relation.

Rappelons la notation $N(V, V')$ (§ 9, p. 38). Nous aurons alors

$$N(a_k^q, b_i^{p-q}) = 0,$$

si i n'est pas égal à k, et

$$N(a_i^q, b_i^{p-q}) = \pm 1.$$

Il reste à voir si l'on doit prendre le signe + ou le signe —.

Pour nous en rendre compte, considérons deux éléments correspondants de P et de P' que j'appellerai a_i^q et b_i^{p-q} ; considérons d'autre part deux éléments correspondants a_j^{q-1} et b_j^{p-q+1} de telle façon que a_j^{q-1} appartienne à a_i^q, et b_i^{p-q} à b_j^{p-q+1}.

Je pourrai toujours choisir mes coordonnées de telle façon que les équations de a_i^q soient

$$(3) \quad F_1 = F_2 = \ldots = F_{p-q} = 0,$$

les F étant des fonctions de coordonnées $y_1, y_2, \ldots, y_p$ qui définiront la position d'un point sur la variété V.

Soient de même

$$(4) \quad \phi_1 = \phi_2 = \ldots = \phi_{q-1} = 0$$

les équations de b_j^{p-q+1} ; je pourrai alors supposer que les équations de a_j^{q-1} s'obtiennent en adjoignant aux équations (1) l'équation $\psi = 0$, et que celles de b_i^{p-q} s'obtiennent en adjoignant aux équations (2) l'équation $\psi = 1$. Je pourrai m'arranger pour que *la même* fonction ψ figure au premier membre de ces deux équations.

Alors parmi les inégalités, qui avec les égalités (1) complètent la définition de a_i^q, devra figurer l'inégalité

$$\psi > 0.$$

De même parmi les inégalités, qui avec les égalités (2) complètent la définition de b_j^{p-q+1}, devra figurer l'inégalité

$$\psi < 1.$$

Si nous voulons que ϵ_{ij}^q soit égal à +1, il faut d'après nos conventions que les équations de a_j^{q-1} se mettent dans l'ordre suivant :—

$$F_1 = F_2 = \ldots = F_{p-q} = \psi = 0 ;$$

et si nous voulons en même temps que $\epsilon_{ji}^{\prime p-q+1} = +1$, il faut que les équations de b_i^{p-q} se mettent dans l'ordre suivant :—

$$\phi_1 = \phi_2 = \ldots = \phi_{q-1} = 1 - \psi = 0.$$

Le nombre $N(a_i^q, b_i^{p-q})$ dépend du signe du déterminant fonctionnel de

$$F_1, F_2, \ldots, F_{p-q}, \phi_1, \phi_2, \ldots, \phi_{q-1}, 1-\psi.$$

De même le nombre $N(a_j^{q-1}, b_j^{p-q+1})$ dépend du signe du déterminant fonctionnel de

$$F_1, F_2, \ldots, F_{p-q}, \psi, \phi_1, \phi_2, \ldots, \phi_{q-1}.$$

Nous pouvons toujours supposer que les fonctions F, ϕ et ψ aient été choisies de telle sorte que ces déterminants ne s'annulent pas dans le domaine considéré.

Nous voyons alors que les deux déterminants sont de même signe si q est pair, et de signe contraire si q est impair.

Nous aurons dans le premier cas

$$N(a_i^q, b_i^{p-q}) = N(a^{q-1}, b_j^{p-q+1}),$$

et dans le second cas

$$N(a_i^q, b_i^{p-q}) = -N(a_j^{q-1}, b_j^{p-q+1}).$$

Comme nous pourrons toujours supposer

$$N(a_i^0, b_i^p) = +1,$$

nous trouverons successivement

$$N(a_i^1, b_i^{p-1}) = -1, \quad N(a_i^2, b_i^{p-2}) = -1, \quad N(a_i^3, b_i^{p-3}) = +1,$$
$$N(a_i^4, b_i^{p-4}) = +1, \quad \ldots.$$

La seule chose à retenir, c'est que le nombre $N(a_i^q, b_i^{p-q})$ ne dépend que de q.

Cela posé, on peut former avec les nombres ϵ_{ij}^q un tableau que j'appellerai T_q, et où le nombre ϵ_{ij}^q occupera la i^e ligne et la j^e colonne. Dans ce tableau T_q il y aura donc autant de lignes que de a_i^q et de colonnes que de a_j^{q-1}.

J'ai appelé α_q le nombre des a_i^q de sorte que le tableau T_q aura α_q lignes et α_{q-1} colonnes. En particulier, le tableau T_1 nous donnera la relation entre les arêtes et les sommets, le tableau T_2 entre les faces et les arêtes, etc.

J'appellerai T'_q le tableau qui est formé avec P', comme T_q avec P. Nous voyons que le tableau T'_q s'obtient en partant du tableau T_{p-q+1}, *permutant* les lignes avec les colonnes, et réciproquement.

Nous avons désigné (c, § 3, p. 14) par $\alpha_q-\alpha'_q$ le nombre des homologies distinctes entre les a_i^q, et par $\alpha_q-\alpha''_q$ le nombre des congruences distinctes entre les a_i^q (les a_j^{q-1} étant éliminés) ; par

$$P_q=\alpha'_q-\alpha''_q+1$$

le nombre de Betti correspondant aux a_i^q.

Nous avons appelé β_q, β'_q et β''_q les nombres analogues à α_q, α'_q et α'', et se rapportant au polyèdre P', de telle sorte que

$$\beta_q=\alpha_{p-q}.$$

2. *Réduction des Tableaux.*

Considérons un tableau T formé de nombres entiers rangés en un certain nombre de lignes et de colonnes. Tels sont nos tableaux T_q.

Supposons que l'on puisse faire sur ce tableau les opérations suivantes :—

1° Ajouter une colonne à une autre ou l'en retrancher;

2° Permuter deux colonnes et changer le signe de l'une d'elles ;

3° Faire les mêmes opérations sur les lignes.

En combinant ces opérations, on pourra faire subir aux colonnes une substitution linéaire quelconque pourvu que les coëfficients soient entiers et le déterminant égal à 1. De même pour les lignes.

Quel est, par le moyen de ces opérations, le plus grand degré de simplicité auquel on puisse réduire un tableau ?—C'est ce que nous allons examiner.

Supposons d'abord, pour fixer les idées, que le tableau T n'ait pas plus de lignes que de colonnes.

Lemme I.—Soit

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{vmatrix}$$

un tableau T que je suppose, pour fixer les idées, de trois lignes et de cinq colonnes.

Je suppose que les quinze nombres a, b, c soient premiers entre eux ; je dis qu'on pourra toujours trouver trois nombres α_1, β_1, γ_1, tels que les cinq nombres

$$h_{1i}=\alpha_1a_i+\beta_1b_i+\gamma_1c_i \quad (i=1,\,2,\,3,\,4,\,5)$$

soient premiers entre eux.

Pour cela les nombres α_1, β_1, γ_1 doivent d'abord remplir une première condition : ils doivent être premiers entre eux. Si cette condition est remplie, on pourra trouver six autres nombres

$$\alpha_2,\ \beta_2,\ \gamma_2\ ;\ \alpha_3,\ \beta_3,\ \gamma_3,$$

tels que le déterminant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 1.$$

Posons alors

$$h_{ki} = \alpha_k a_i + \beta_k b_i + \gamma_k c_i \quad (i = 1, 2, 3, 4, 5\ ;\ k = 1, 2, 3).$$

Soit

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\ ;$$

la règle de la multiplication des déterminants nous donnera

$$\begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta.$$

Ce qui montre que le plus grand commun diviseur des trois nombres h_{11}, h_{12}, h_{13}, et par conséquent celui des cinq nombres h_{1i}, divise Δ. Il doit diviser de même tous les déterminants obtenus en supprimant deux colonnes dans le tableau, et par conséquent le plus grand commun diviseur, M, de tous ces déterminants.

Soit p un facteur premier quelconque de M. Comme nos quinze nombres a, b, c sont premiers entre eux, l'un d'eux au moins, par exemple c_5, ne sera pas divisible par p.

Si nous prenons alors

$$(1) \quad \alpha_1 \equiv 0, \quad \beta_1 \equiv 0, \quad \gamma_1 \equiv c_5^{p-2} \pmod{p},$$

il viendra

$$h_{15} \equiv c_5^{p-1} \equiv 1 \pmod{p},$$

de sorte que le plus grand commun diviseur des cinq nombres h_{1i} ne sera pas divisible par p.

Nous obtiendrons un système de congruences analogues à (1) pour chacun des facteurs premiers de M. On pourra satisfaire à la fois à toutes ces congruences puisqu'elles ont lieu par rapport à des modules premiers différents.

Alors le plus grand commun diviseur des cinq nombres h_{1i} ne sera divisible par aucun des facteurs premiers de M ; et, comme il doit diviser M, il sera égal à 1.

1$^{\text{er}}$ *Corollaire.*—Si l'on fait subir aux lignes du tableau la substitution linéaire

$$\begin{array}{ccc} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3, \end{array}$$

il est clair que les éléments de la i^{e} colonne qui étaient

$$a_i, \quad b_i, \quad c_i$$

deviendront $$h_{1i}, \quad h_{2i}, \quad h_{3i},$$

d'où cette conséquence :

Si les éléments du tableau sont premiers entre eux, on peut réduire le tableau de telle sorte que les éléments de la première ligne soient premiers entre eux.

2$^{\text{e}}$ *Corollaire.*—Si les éléments du tableau ont pour plus grand commun diviseur δ, on peut réduire le tableau de telle sorte que les éléments de la première ligne aient pour plus grand commun diviseur δ.

Théorème.—Soit m le nombre des colonnes et n celui des lignes ($m \geqq n$) ; soit M_0 le plus grand commun diviseur de tous les déterminants obtenus en supprimant dans le tableau $m-n$ colonnes quelconques ; soit M_1 le plus grand commun diviseur de tous les déterminants obtenus en supprimant dans le tableau $m-n+1$ colonnes et une ligne ; soit M_2 celui des déterminants obtenus en supprimant $m-n+2$ colonnes et deux lignes, etc. ; soit enfin M_{n-1} celui des déterminants obtenus en supprimant $m-1$ colonnes et $n-1$ lignes, c'est-à-dire en d'autres termes celui de tous les éléments.

Ces nombres $M_0, M_1, \ldots, M_{n-1}$ ne seront pas altérés par les opérations faites soit sur les lignes, soit sur les colonnes.

Il va sans dire que le nombre M_k devrait être considéré comme nul si tous les déterminants correspondants étaient nuls.

Nous pourrons alors énoncer notre corollaire sous la forme suivante :—

3$^{\text{e}}$ *Corollaire.*—On peut réduire le tableau de telle sorte que le plus grand commun diviseur des éléments de la première ligne soit M_{n-1}.

Lemme II.—On peut, par une transformation entre les colonnes, réduire le tableau de telle sorte que le premier élément de la première ligne devienne M_{n-1}, et que tous les autres éléments de la première ligne deviennent nuls.

Nous allons faire subir, en effet, aux colonnes (supposées comme plus haut au nombre de $m = 5$) la substitution linéaire

$$(2) \quad \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_1 & \dots & & \dots & \beta_5 \\ \gamma_1 & \dots & & \dots & \gamma_5 \\ \delta_1 & \dots & & \dots & \delta_5 \\ \zeta_1 & \dots & & \dots & \zeta_5 \end{vmatrix},$$

dont le déterminant doit être égal à 1. Soient

$$a_1, \; a_2, \; a_3, \; a_4, \; a_5$$

les éléments de la première ligne. Après les réductions que le tableau a déjà subies, le plus grand commun diviseur de ces cinq nombres est devenu M_{n-1}. Nous pouvons alors choisir la substitution (2) de telle sorte que l'on ait

$$\Sigma \alpha_i a_i = M_{n-1}, \quad \Sigma \beta_i a_i = \Sigma \gamma_i a_i = \Sigma \delta_i a_i = \Sigma \zeta_i a_i = 0.$$

Alors, après la transformation, les éléments de la première ligne seront

$$M_{n-1}, \; 0, \; 0, \; 0, \; 0.$$

Lemme III.—Je dis maintenant qu'on peut, par une transformation entre les lignes, réduire à zéro tous les éléments de la première colonne, sauf le premier qui reste égal à M_{n-1}.

En effet, après les réductions déjà faites, les éléments de la première colonne (supposés au nombre de $n = 3$) sont

$$M_{n-1}, \; q_2 M_{n-1}, \; q_3 M_{n-1},$$

q_2 et q_3 étant des entiers ; et en effet, d'après nos hypothèses, tous nos éléments sont divisibles par M_{n-1}.

Si alors nous retranchons de la seconde ligne la première ligne multipliée par q_2, et de la troisième ligne la première ligne multipliée par q_3, la première colonne devient

$$M_{n-1}, \; 0, \; 0.$$

D'ailleurs la première ligne ne change pas.

Si l'on supprimait maintenant la première ligne et la première colonne du tableau T, il resterait un tableau T' de $m-1$ colonnes et de $n-1$ lignes, par rapport auquel les nombres

$$\frac{M_0}{M_{n-1}}, \; \frac{M_1}{M_{n-1}}, \; \dots$$

joueraient le même rôle que les nombres M_0, M_1, ... par rapport au tableau T.

En particulier,

le plus grand commun diviseur des éléments de T' est $\frac{M_{n-2}}{M_{n-1}}$.

Nous pouvons maintenant continuer la réduction, mais en opérant seulement sur les $m-1$ dernières colonnes et sur les $n-1$ dernières lignes. La première ligne ne changera plus puisque ses $m-1$ derniers éléments sont nuls, ni la première colonne non plus puisque ses $n-1$ derniers éléments sont nuls.

On pourra opérer sur le tableau T' comme nous avons opéré sur le tableau T. Après cette nouvelle réduction :

1° Tous les éléments de la première ligne et ceux de la première colonne sont restés nuls, sauf le premier élément de la première ligne et de la première colonne qui est resté égal à M_{n-1}.

2° Tous les éléments de la seconde ligne et ceux de la seconde colonne sont devenus nuls, sauf le second élément de la seconde ligne et de la seconde colonne qui est devenu $\frac{M_{n-2}}{M_{n-1}}$.

3° Si on supprime les deux premières lignes et les deux premières colonnes, on obtient un tableau T'' de $m-2$ colonnes et de $n-2$ lignes, par rapport auquel les nombres

$$\frac{M_0}{M_{n-2}}, \frac{M_1}{M_{n-2}}, \dots, \frac{M_{n-3}}{M_{n-2}}$$

jouent le même rôle que les nombres M_0, M_1, ..., M_{n-1} par rapport au tableau T. Et ainsi de suite.

A la fin de la réduction, l'élément qui appartient à la i^e ligne et à la j^e colonne est nul si i n'est pas égal à j ; l'élément qui appartient à la i^e ligne et à la i^e colonne est égal à $\frac{M_{n-i}}{M_{n-i+1}}$.

Les n nombres

$$(3) \quad M_{n-1}, \frac{M_{n-2}}{M_{n-1}}, \frac{M_{n-3}}{M_{n-2}}, \dots, \frac{M_1}{M_2}, \frac{M_0}{M_1}$$

peuvent s'appeler les *invariants* du tableau T.

On peut remarquer :

1° Que chacun de ces invariants divise le suivant ;

2° Que quelques-uns de ces invariants peuvent être nuls, mais que, si l'un d'eux l'est, tous ceux qui le suivent le sont également.

Si le tableau T avait plus de lignes que de colonnes, la réduction se ferait de la même manière, seulement il faudrait intervertir le rôle des lignes et des colonnes.

On aurait alors $m<n$; le nombre M_0 serait le plus grand commun diviseur des déterminants obtenus en supprimant $n-m$ lignes ; en général, M_i serait le plus grand commun diviseur des déterminants obtenus en supprimant $n-m+i$ lignes et i colonnes quelconques. Enfin, le plus grand commun diviseur des éléments du tableau T serait M_{m-1}.

En général, le nombre des invariants serait le plus petit des deux nombres n et m.

3. *Comparaison des Tableaux T_q et T'_q.*

Le tableau T_q nous fait connaître les relations entre les a_i^q et les a_j^{q-1} dans le polyèdre P. A chaque ligne de ce tableau correspond un a_i^q et à chaque colonne un a_j^{q-1}. A chaque ligne de ce tableau correspond également une congruence

$$(1) \quad a_i^q \equiv \Sigma \epsilon_{ij}^q a_j^{q-1}$$

entre les a_i^q et les a_j^{q-1} et une homologie

$$(2) \quad \Sigma \epsilon_{ij}^q a^{q-1} \sim 0$$

entre les a_j^{q-1}.

Qu'arrivera-t-il maintenant si, par les opérations du paragraphe précédent, on réduit le tableau T_q ?—A chaque ligne du tableau réduit correspondra une combinaison linéaire des a_i^q, à chaque colonne une combinaison linéaire des a_j^{q-1}. J'ai expliqué (c, § 8, p. 41) d'après quelles règles ces combinaisons linéaires doivent être formées. Voici comment ces règles peuvent être résumées.

Supposons que, pour passer du tableau T_q au tableau réduit, on applique aux lignes de T_q une certaine substitution linéaire S, et aux colonnes une autre substitution linéaire σ. Soit σ' la substitution contragrédiente de σ (je veux dire que, si l'on a deux séries de α_{q-1} variables x_i et y_i, et qu'on applique la substitution σ à la première série et la substitution σ' à la seconde, la forme $\Sigma x_i y_i$ ne devra pas être altérée).

Supposons alors que S change a_i^q en

$$c_i^q = \sum_{j=1}^{\alpha_q} \lambda_{ij} a_j^q,$$

et que σ' change a_i^{q-1} en

$$d_i^{q-1} = \sum_{i=1}^{\alpha_{q-1}} \mu_{ij} a_j^{q-1}.$$

Nous ferons correspondre à la i^e ligne du tableau réduit la combinaison linéaire c_i^q, et à la i^e colonne la combinaison linéaire d_i^{q-1}.

Dans notre tableau réduit, tous les éléments sont nuls, sauf ceux de la i^e ligne et de la i^e colonne, qui sont donnés d'après le paragraphe précédent par la formule

$$\frac{M_{n-i}}{M_{n-i-1}}.$$

Je désignerai, pour abréger, par ω_i^q cet élément de la i^e ligne et de la i^e colonne; et je conviendrai que ω_i^q doit être regardé comme nul, si i est plus grand que le plus petit des deux nombres α_q et α_{q-1} (nombre des lignes et nombre des colonnes).

A la i^e ligne du tableau réduit correspondra alors la congruence

$$\text{(1 bis)} \quad c_i^q \equiv \omega_i^q d_i^{q-1}$$

et l'homologie

$$\text{(2 bis)} \quad \omega_i^q d_i^{q-1} \sim 0.$$

Les congruences et les homologies (1 bis) et (2 bis) peuvent se déduire des congruences et homologies (1) et (2) par addition, soustraction, multiplication, *mais sans division*, et réciproquement.

Si $\alpha_{q-1} > \alpha_q$, et si $i > \alpha_q$, ω_i^q est nul, de sorte que la congruence et l'homologie (1 bis) et (2 bis) se réduisent à

$$c_i^q \equiv 0 \quad \text{et} \quad 0 \sim 0.$$

Les nombres ω_i^q sont ce que j'ai appelé dans le paragraphe précédent les *invariants* du tableau T_q. Supposons que *parmi ces invariants il y en ait* γ_q *qui ne soient pas nuls;* on aura, bien entendu,

$$\gamma_q \leqq \alpha_q, \quad \gamma_q \leqq \alpha_{q-1}.$$

Parmi les congruences (1 bis), les γ_q premières contiendront à la fois c_i^q et d_i^{q-1} puisque ω_i^q ne sera pas nul. Au contraire, les $\alpha_q - \gamma_q$ dernières s'écriront

$$c_i^q \equiv 0,$$

et ne contiendront plus les a_i^{q-1}; il est clair que toutes ces congruences sont distinctes, et qu'on obtient ainsi toutes les congruences entre les a_i^q d'où les a_j^{q-1} sont éliminés. On aura donc

$$\alpha_q - \alpha_q'' = \alpha_q - \gamma_q, \quad \alpha_q'' = \gamma_q.$$

Maintenant, parmi les homologies (2 bis), les $\alpha_q - \gamma_q$ dernières se réduisent à des identités, mais les γ_q premières sont distinctes; on a donc

$$\alpha_{q-1} - \alpha_{q-1}' = \gamma_q,$$

d'où pour le nombre de Betti

$$P_q = a_q - \gamma_{q+1} - \gamma_q + 1.$$

Comparons maintenant le tableau T_q au tableau correspondant T'_{p-q+1} relatif au polyèdre réciproque P'. Ce tableau, qui se déduit de T_q en permutant les lignes avec les colonnes, a $\beta_{p-q+1} = a_{q-1}$ lignes et $\beta_{p-q} = a_q$ colonnes. Le nombre γ_q est le même pour les deux tableaux, de sorte qu'il vient

$$\beta''_{p-q+1} = \gamma_q = a''_q, \quad \beta_{p-q} - \beta'_{p-q} = \gamma_q,$$

$$\beta'_{p-q} = \beta_{p-q} - \gamma_q = a_q - a''_q\,;$$

d'où

$$\beta'_{p-q+1} = a_{q-1} - \gamma_{q-1},$$

et pour le nombre de Betti P'_{p-q+1} relatif au polyèdre P'

$$P'_{p-q+1} = \beta'_{p-q+1} - \beta''_{p-q+1} + 1 = a_{q-1} - \gamma_{q-1} - \gamma_q + 1.$$

Nous déduisons de là

$$P'_{p-q} = P_q,$$

ce qui, si l'on se rappelle que les nombres de Betti relatifs aux deux polyèdres réciproques P et P' sont les mêmes, montre que les nombres de Betti également distants des extrêmes sont égaux.

Revenons aux homologies (2 bis). Si l'on admet que l'on a le droit de diviser les homologies par un entier différent de zéro, les γ_q premières homologies nous donneront

$$d_i^{q-1} \sim 0 \quad (i = 1, 2, \ldots, \gamma_q),$$

et la plus générale des homologies entre les a_j^{q-1} s'écrira

$$(3) \qquad \sum_{i=1}^{\gamma_q} \lambda_i d_i^{q-1} \sim 0,$$

les λ_i étant des entiers quelconques. Si, au contraire, on n'admet pas que l'on ait le droit de diviser les homologies, l'homologie la plus générale s'écrira

$$(4) \qquad \sum_{i=1}^{\gamma_q} \lambda_i \omega_i^q d_i^{q-1} \sim 0,$$

les λ_i étant des entiers. Pour que les deux définitions des nombres de Betti (c, § 1, p. 2) coincident, il faut, et il suffit, que les deux formules (3) et (4) concordent, c'est-à-dire que tous les invariants ω_i^q qui ne sont pas nuls soient égaux à 1 (c, § 9, p. 48).

Envisageons maintenant les combinaisons linéaires des a_j^{q-1} qui seraient homologues à zéro en vertu des homologies (3), et demandons-*nous quelles* sont parmi ces combinaisons celles qui restent distinctes, si, abandonnant les homologies (3), on se borne aux homologies (4) sans admettre le droit de diviser les homologies.

Nous verrons tout de suite que le nombre de ces expressions qui sont ainsi distinctes est précisément le produit

$$\omega_1^q \, \omega_2^q \, \dots \, \omega_{\gamma_q}^q.$$

Or, en se reportant aux notations du paragraphe précédent, on voit que ce produit n'est autre chose que l'un des nombres de la suite

$$M_0, \; M_1, \; M_2, \; \dots,$$

et précisément le premier nombre de cette suite qui n'est pas nul (c. § 9, p. 48).

Ce qui précède montre combien il importe de distinguer deux sortes de variétés.

Celles de la première sorte, que j'appellerai *variétés sans torsion*, seront celles pour lesquelles les invariants de tous les tableaux T_q sont tous égaux à 0 ou à 1; pour lesquelles, par conséquent, les deux formules (3) et (4) concordent et les deux définitions des nombres de Betti sont d'accord.

Celles de la seconde sorte, que j'appellerai *variétés à torsion*, seront celles pour lesquelles certains de ces invariants ne sont égaux ni à 0, ni à 1, et pour lesquelles, par conséquent, les deux définitions des nombres de Betti ne sont pas d'accord. Dans ce cas nous adopterons toujours, sauf avis contraire, la seconde définition (c, § 1).

Cette dénomination se justifie parce que la présence d'invariants plus grands que 1 est due, comme nous le verrons plus loin, à une circonstance assimilable à une véritable torsion de la variété sur elle-même.

4. *Application à quelques Exemples.*

Désireux d'appliquer ce qui précède aux exemples signalés dans l'"Analysis Situs" (p. 49, sqq.), je dois d'abord faire une distinction entre plusieurs sortes de polyèdres.

Les polyèdres ordinaires ou de la première sorte seront ceux dont tous les a_i^q sont des polyèdres simplement connexes (homéomorphes à des hypersphères) et tels que tous les éléments de ces a_i^q soient distincts; par exemple, dans l'espace ordinaire, le tétraèdre sera un polyèdre de la première sorte parce qu'il admet quatre faces qui sont des triangles et, par conséquent, des polygones simplement connexes (homéomorphes à des cercles), et que chacun de ces triangles a ses trois côtés distincts de même que ses trois sommets.

Les polyèdres de la seconde sorte seront ceux dont tous les a_i^q seront des polyèdres simplement connexes, mais tels que tous les éléments de ces a_i^q ne soient pas distincts. Soit, par exemple, dans l'espace ordinaire un tore; par un point A de la surface de ce tore menons un méridien et un parallèle. Ces deux coupures ne diviseront pas la surface du tore en deux régions, mais elles la rendront simplement connexe. Cette surface ainsi rendue simplement connexe sera homéomorphe à un rectangle, dont deux côtés opposés correspondraient aux deux lèvres de la coupure méridienne et les deux autres côtés aux deux lèvres de la coupure parallèle. Le tore forme ainsi une espèce de polyèdre qui n'a qu'une seule face; cette face est un quadrilatère; elle est donc simplement connexe; mais les quatre côtés de ce quadrilatère ne sont pas distincts, deux se confondent avec la coupure méridienne et deux avec la coupure parallèle; de même les quatre sommets ne sont pas distincts puisqu'ils se confondent tous les quatre avec le point A. Le polyèdre ainsi défini est donc un polyèdre de la seconde sorte.

Enfin, les polyèdres de la troisième sorte seront ceux dont tous les a_i^q ne sont pas simplement connexes.

Les propriétés des polyèdres de la première sorte s'étendent pour la plupart à ceux de la seconde sorte. Observons toutefois une différence. Dans un polyèdre de la première sorte, toute a_j^{p-1} sépare l'une de l'autre deux a_i^p, et n'appartient à aucun autre a_i^p. Par conséquent, dans chaque colonne du tableau T_p il y aura un des nombres ϵ_{ij}^p qui sera égal à $+1$, un autre à -1, et tous les autres à 0.

Il n'en est plus de même avec les polyèdres de la seconde sorte. Il peut arriver que deux des a_j^{p-1} d'une même a_i^p ne soient pas distinctes. Dans ce cas, après avoir franchi cette a_j^{p-1}, on se retrouvera dans cette même a_i^p où l'on était déjà avant de l'avoir passée. Ainsi pour reprendre notre tore de tout à l'heure, qui était un polyèdre à une seule face: après avoir passé la coupure méridienne, par exemple, on se retrouvera toujours dans cette même et unique face où l'on était avant le passage. Il arrive alors que cette a_j^{p-1} n'a de relation qu'avec cette a_i^p; et de plus, elle est deux fois en relation avec cette même a_i^p, une fois en relation directe, une autre fois en relation inverse, de sorte que les deux relations se compensant, le nombre ϵ_{ij}^p correspondant est égal à zéro. Dans ce cas, tous les nombres ϵ_{ij}^p qui figurent dans la colonne correspondante du tableau T_p sont nuls.

Dans les exemples en question (p. 49 sqq.), les variétés fermées à trois dimensions que l'on envisage peuvent être regardées comme des polyèdres de la seconde sorte. Chacun de ces polyèdres a une seule case (qui dans les premier, troisième et quatrième exemples est un cube, et dans le cinquième un octaèdre), mais les faces de cette case se confondent deux à deux.

1er *Exemple.*—

1ère face $ABDC = A'B'D'C'$, 1ère arête $AB = CD = A'B' = C'D'$;
2e „ $ACC'A' = BDD'A'$, 2e „ $AC = BD = A'C' = B'D'$;
3e „ $CDD'C' = ABB'A'$, 3e „ $AA' = BB' = CC' = DD'$;

Une case unique, un sommet unique.

Les trois tableaux T_1, T_2 et T_3 se composent uniquement de zéros. Tous leurs invariants sont donc nuls.

3e *Exemple.*—

1ère face $ABDC = B'D'C'A'$, 1ère arête $AB = B'D' = C'C$;
2e „ $ABB'A' = C'CDD'$, 2e „ $AC = DD' = B'A'$;
3e „ $ACC'A' = DD'B'B$, 3e „ $AA' = C'D' = DB$;
4e „ $CD = BB' = A'C'$;

1er sommet $A = B' = C' = D$, 2e sommet $B = D' = A' = C$.

Tableau T_3.

$$\begin{vmatrix} 0 & 0 & 0 \end{vmatrix}.$$

Tableau T_2.

$$\begin{vmatrix} +1 & -1 & -1 & -1 \\ +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & -1 \end{vmatrix}.$$

Tableau T_1.

$$\begin{vmatrix} +1 & -1 \\ +1 & -1 \\ +1 & -1 \\ -1 & +1 \end{vmatrix}.$$

Le tableau T_3 n'a qu'un invariant qui est nul; le tableau T_1 en a deux qui sont 0 et 1; le tableau T_2 en a trois qui sont

$$1,\ 2 \text{ et } 2.$$

4e *Exemple.*—

1ère face $ABDC = B'D'C'A'$, 1ère arête $AA' = CC' = BB' = DD'$;
2e „ $ABB'A' = CDD'C'$, 2e „ $AB = CD = B'D' = A'C'$;
3e „ $ACC'A' = BDD'B'$, 3e „ $AC = BD = D'C' = B'A'$;

Une case unique, un sommet unique.

Tableau T_3. $\quad$ Tableau T_2. $\quad$ Tableau T_1.

$$\begin{vmatrix} 0 & 0 & 0 \end{vmatrix}. \qquad \begin{vmatrix} 0 & 0 & 0 \\ 0 & +1 & +1 \\ 0 & +1 & -1 \end{vmatrix}. \qquad \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}.$$

Les tableaux T_1 et T_3 n'ont qu'un invariant qui est 0 ; le tableau T_2 en a trois qui sont

$$1,\ 2 \text{ et } 0.$$

5^e *Exemple.*—

1ère face $ABC = FED$,	1ère arête $AB = FE$,	1er sommet $A = F$;
2^e „ $ACE = FDB$,	2^e „ $AC = FD$,	2^e „ $B = E$;
3^e „ $AED = FBC$,	3^e „ $AE = FB$,	3^e „ $C = D$;
4^e „ $ADB = FCE$,	4^e „ $AD = FC$;	
	5^e „ $BC = ED$;	
	6^e „ $CE = DB$.	

Tableau T_3. $\quad$ Tableau T_2. $\quad$ Tableau T_1.

$$\begin{vmatrix} 0 & 0 & 0 & 0 \end{vmatrix}. \qquad \begin{vmatrix} +1 & -1 & 0 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 & 0 & +1 \\ 0 & 0 & +1 & -1 & +1 & 0 \\ -1 & 0 & 0 & +1 & 0 & +1 \end{vmatrix}. \qquad \begin{vmatrix} +1 & -1 & 0 \\ +1 & 0 & -1 \\ +1 & -1 & 0 \\ +1 & -1 & 0 \\ 0 & +1 & -1 \\ 0 & -1 & +1 \end{vmatrix}.$$

Les invariants sont :

$$0 \text{ pour } T_3; \quad 2,\ 1,\ 1 \text{ et } 1 \text{ pour } T_2; \quad 0,\ 1 \text{ et } 1 \text{ pour } T_1.$$

Passons maintenant au sixième exemple (p. 57).

Ainsi qu'on l'a vu (§ 14, p. 71), les équivalences fondamentales s'écrivent

$$\begin{aligned} C_1 + C_2 &\equiv C_2 + C_1, \\ C_1 + C_3 &\equiv C_3 + \alpha C_1 + \gamma C_2, \\ C_2 + C_3 &\equiv C_3 + \beta C_1 + \delta C_2. \end{aligned}$$

Pour écrire les homologies qui peuvent se déduire des homologies fondamentales par addition et multiplication, *mais sans division*, il suffit de se donner le droit d'intervertir l'ordre des termes dans les deux membres de ces équivalences fondamentales ; on trouve ainsi

$$0 \sim 0; \quad (\alpha - 1)\,C_1 + \gamma C_2 \sim 0; \quad \beta C_1 + (\delta - 1)\,C_2 \sim 0.$$

Le déterminant $$(\alpha-1)(\delta-1)-\beta\gamma$$
est égal à $$2-\alpha-\delta.$$

Soit, d'autre part, μ le plus grand commun diviseur des quatre nombres
$$\alpha-1,\ \delta-1,\ \beta,\ \gamma\,;$$
l'examen des homologies que nous venons d'écrire montre que les deux invariants du tableau T_2 qui ne sont pas égaux à 0 ou à 1 sont égaux à
$$\mu \quad \text{et} \quad \frac{2-\alpha-\delta}{\mu}.$$
(Le nombre μ peut d'ailleurs être égal à 1.)

Quant aux invariants des tableaux T_1 et T_2, ils sont toujours tous, comme nous le verrons plus loin, égaux à 0 ou à 1.

Soit, par exemple,
$$\alpha=-1,\ \ \beta=1,\ \ \gamma=-1,\ \ \delta=0.$$
On a $$\mu=1,\ \ 2-\alpha-\delta=3,$$
de sorte que l'un de nos invariants est égal à 3 et l'autre à 1.

Cela peut d'ailleurs se vérifier en formant le tableau T_2. Soient
$$(x+1,\ y,\ z),$$
$$(x,\ y+1,\ z),$$
$$(-x+y,\ -x,\ z+1)$$
les trois substitutions du groupe G, que j'appellerai S_1, S_2 et S_3, et qui correspondront aux trois contours fondamentaux C_1, C_2, C_3 (§ 13, p. 68).

La variété étudiée peut être regardée comme engendrée par le cube $ABCDA'B'C'D'$ (§ 10, p. 49). Seulement la face $ABCD$ devra être considérée comme décomposée en deux triangles ABD et ACD, de même que la face $A'B'C'D'$ en deux triangles $D'A'B'$ et $C'D'A'$.

Il est aisé de voir que la face $ABB'A'$ est changée en $CDD'C'$ par la substitution S_2, la face $ACC'A'$ en $BDD'B'$ par la substitution S_1, la face ABD en $D'A'B'$ par la substitution $S_3S_1S_2$, la face ACD en $C'D'A'$ par la substitution S_3S_2.

Notre polyèdre a donc :

1° Une seule case ;

2° Quatre faces, à savoir :

$$\begin{array}{lll} 1^{\text{ère}} & \text{face} & ABB'A'=CDD'C', \\ 2^{\text{e}} & ,, & ACC'A'=BDD'B', \\ 3^{\text{e}} & ,, & ABD=D'A'B', \\ 4^{\text{e}} & ,, & ACD=C'D'A'\,; \end{array}$$

3° Quatre arêtes, à savoir :

1$^{\text{ère}}$ arête $AA' = BB' = CC' = DD'$,
2$^{\text{e}}$ „ $AB = CD = D'A'$,
3$^{\text{e}}$ „ $AC = BD = C'D' = A'B'$,
4$^{\text{e}}$ „ $AD = C'A' = D'B'$;

4° Un seul sommet.

Les tableaux T_1 et T_3 sont entièrement composés de zéros.

Voici le tableau T_2 :

$$\begin{vmatrix} 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & +1 \\ 0 & +1 & +1 & -1 \\ 0 & +1 & +1 & -1 \end{vmatrix}.$$

On voit que les invariants de ce tableau sont

$$1,\ 1,\ 3,\ 0.$$

Passons maintenant à l'exemple de M. Heegaard. Soient x_1, x_2, y_1, y_2, z_1, z_2 les coordonnées d'un point dans l'espace à six dimensions ; soit

$$x = x_1 + x_2\sqrt{-1} = |x|\, e^{\xi\sqrt{-1}},$$
$$y = y_1 + y_2\sqrt{-1} = |y|\, e^{\eta\sqrt{-1}},$$
$$z = z_1 + z_2\sqrt{-1} = |z|\, e^{\zeta\sqrt{-1}}.$$

Notre variété aura pour équations

$$z^2 = xy, \quad x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1,$$

d'où

$$|z^2| = |xy|, \quad \zeta = \frac{\xi + \eta}{2}, \quad |x^2| + |y^2| = 1.$$

Pour obtenir la variété tout entière, il faut que nous fassions varier

1° $|x|$ de 0 à 1, ce qui fait varier en même temps $|y|$ de 1 à 0 ;

2° η de 0 à 2π ;

3° $\xi + \eta$ de 0 à 4π.

Le polyèdre ainsi obtenu a une seule case définie par les inégalités

$$0 < |x| < 1, \quad 0 < \eta < 2\pi, \quad 0 < \xi + \eta < 4\pi.$$

Il a deux faces définies par les relations suivantes :—

1$^{\text{ère}}$ face $\qquad \eta = 0, \quad 0 < |x| < 1, \quad 0 < \xi < 4\pi$;

cette face est identique à la suivante :—

$$\eta = 2\pi, \quad 0 < |x| < 1, \quad -2\pi < \xi < 2\pi.$$

2ᵉ face $\xi+\eta=0,\quad 0<|x|<1,\quad 0<\eta<2\pi;$

cette face est identique à la suivante :—

$$\xi+\eta=4\pi,\quad 0<|x|<1,\quad 0<\eta<2\pi.$$

Il a trois arêtes définies par les relations suivantes :—

1ʳᵉ arête $\xi=\eta=0,\quad 0<|x|<1;$

cette arête est identique aux trois suivantes :—

$$\xi=0,\quad \eta=2\pi,\quad 0<|x|<1;$$
$$\xi=-2\pi,\quad \eta=2\pi,\quad 0<|x|<1;$$
$$\xi=\eta=2\pi,\quad 0<|x|<1;$$

2ᵉ arête $x_1=x_2=0,\quad 0<\eta<2\pi;$

3ᵉ „ $y_1=y_2=0,\quad -2\pi<\xi<0,$ identique aux deux suivantes :—

$$y_1=y_2=0,\quad 0<\xi<2\pi;$$
$$y_1=y_2=0,\quad 2\pi<\xi<4\pi.$$

Il a enfin deux sommets, à savoir :

1ᵉʳ sommet $x_1=x_2=0,\quad \eta=0,$ identique au suivant :—

$$x_1=x_2=0,\quad \eta=2\pi;$$

2ᵉ „ $y_1=y_2=0,\quad \xi=-2\pi,$ identique aux trois suivants :—

$$y_1=y_2=0,\quad \xi=0;$$
$$y_1=y_2=0,\quad \xi=2\pi;$$
$$y_1=y_2=0,\quad \xi=4\pi.$$

Le tableau T_3 est entièrement composé de zéros; quant aux tableaux T_1 et T_2, ils s'écrivent

$$T_2=\begin{vmatrix}0&0&+2\\0&1&1\end{vmatrix};\qquad T_1=\begin{vmatrix}+0&-1\\0&0\\0&0\end{vmatrix}.$$

On voit que les invariants sont 0, 2 et 1 pour T_2, 0 et 1 pour T_1.

5. *Extension au cas général d'un Théorème du Premier Complément.*

Je voudrais revenir sur l'une des questions traitées dans un des mémoires antérieurs (C, § 10). Je n'ai envisagé dans l'endroit cité que le cas de $p=3$, et je voudrais faire voir comment on peut étendre les mêmes raisonnements au cas général. Voici de quoi il s'agit :

Soient deux polyèdres réciproques P et P'; considérons d'une part les éléments a_i^q de P, et d'autre part les éléments b_i^q de P'. Je suppose que l'on ait trouvé une congruence

$$(1)\quad \Sigma\lambda_i a_i^q \equiv 0$$

entre les a_i^q ; je dis qu'on pourra faire correspondre à cette congruence une autre congruence entre les b_i^q

$$(2)\quad \Sigma\mu_i b_i^q \equiv 0,$$

et cela de telle sorte que l'on ait l'homologie

$$(3)\quad \Sigma\lambda_i a_i^q \sim \Sigma\mu_i b_i^q.$$

Réciproquement à toute congruence de la forme (2) on pourra faire correspondre une congruence de la forme (1), et cela de telle sorte que les premiers membres de ces deux congruences soient liés par l'homologie (3).

Tel est le théorème qu'il s'agit de démontrer. J'en ai donné une démonstration simple dans le cas de $p = 3$, et il s'agit d'étendre cette démonstration au cas général. Je ferai d'abord une première remarque.

Considérons les congruences

$$(4)\quad a_i^q \equiv \Sigma\epsilon_{ij}^q a_j^{q-1}.$$

Nous savons qu'en les combinant linéairement, on peut en éliminer les a_j^{q-1} et obtenir des congruences de la forme

$$(5)\quad \Sigma\zeta_i a_i^q \equiv 0.$$

Le nombre des congruences distinctes de la forme (5) est celui que nous avons appelé $\alpha_q - \alpha_q''$.

Supposons maintenant que nous considérions les différents éléments a_i^h du polyèdre P, où le nombre h des dimensions doit être plus grand que q, mais peut être égal à $q+1$, $q+2$, …, $p-1$ ou p. Nous donnons une fois pour toutes à ce nombre h une valeur déterminée.

Nous répartirons alors les congruences (4) en groupes, en mettant dans le même groupe deux de ces congruences si les deux a_i^q correspondants appartiennent à un même a_i^h; il est clair qu'il y aura autant de groupes que de a_i^h, et qu'une même congruence pourra se retrouver dans plusieurs groupes, puisque un a_i^q fait partie de plusieurs a_i^h.

En combinant linéairement les congruences (4) *d'un même groupe*, on pourra alors éliminer les a_j^{q-1} et obtenir des congruences de la forme

$$(5\text{ bis})\quad \Sigma\zeta_i' a_i^q \equiv 0.$$

Les congruences (5 bis) font évidemment partie du système des congruences (5), puisque ce dernier système est celui de *toutes* les congruences distinctes de cette forme que l'on peut obtenir par la combinaison des congruences (4). En revanche, il peut y avoir dans le système (5) des congruences qui ne font pas partie du système (5 bis) ; et en effet nous avons obtenu ce dernier système en imposant des restrictions à notre faculté de combiner les congruences (4) puisque nous ne pouvions combiner que celles d'un même groupe.

Je dis d'abord que la congruence (5 bis) entraîne l'homologie

$$(6) \quad \Sigma \zeta_i' a_i^q \sim 0.$$

En effet, la congruence (5 bis) est une congruence entre les éléments du polyèdre a_i^h, et, *comme par hypothèse ce polyèdre est simplement connexe*, cette congruence doit entraîner l'homologie correspondante.

Réciproquement, si l'homologie (6) a lieu, la congruence correspondante fera partie du système (5 bis). En effet, l'homologie (6) ayant lieu entre les éléments du polyèdre a_i^h, doit entraîner la congruence correspondante, et cette congruence doit pouvoir se déduire des congruences fondamentales de la forme (4) *relatives au polyèdre* a_i^h, c'est-à-dire appartenant à un même groupe.

Il résulte de là que le nombre des congruences distinctes du système (5 bis) est $\alpha_q - \alpha_q'$.

Le système (5 bis) reste donc toujours le même quelle que soit la valeur attribuée au nombre h.

Nous voyons en même temps que cette considération permettrait de trouver le nombre de Betti P_q en considérant seulement le tableau T_q, pourvu que l'on sût en outre si deux congruences (4) appartiennent ou non à un même groupe.

Introduisons maintenant une notion qui peut être considérée comme la généralisation de la notion de pyramide. Soit a_q un domaine appartenant à un espace plan P_q à q dimensions ; soit b_m un domaine appartenant à un autre espace plan P_m' à m dimensions. Je supposerai que ces deux espaces plans n'ont aucun point commun. Je pourrai alors par ces deux espaces faire passer un espace plan Π à $q+m+1$ dimensions, et un seul.

Cela posé, joignons par des droites chacun des points du domaine a_q à chacun des points du domaine b_m. L'ensemble de ces droites engendrera un certain domaine appartenant à l'espace plan Π, ayant $q+m+1$ dimensions que je désignerai par la notation $a_q b_m$, et que je pourrai appeler pyramide généralisée rectiligne.

Si, en effet, le domaine a_q se réduisait à un polygone plan ($q = 2$), et le domaine b_m à un point ($m = 0$), le domaine $a_q b_m$ se réduirait à une pyramide ordinaire ayant a_q pour base et b_m pour sommet.

Toute figure homéomorphe à une pyramide généralisée rectiligne pourra s'appeler pyramide généralisée.

Cela posé, envisageons un élément a_i^q du polyèdre P et un élément b_j^m du polyèdre réciproque P'; cet élément b_j^m correspond à un élément a_j^{p-m} du polyèdre P. Je suppose que l'élément a_i^q fasse partie de l'élément a_j^{p-m} ; nous aurons donc

$$q < p - m; \quad p \geqq q + m + 1.$$

Je remarque de plus que tout point de b_j^m fera partie de l'un des a_k^p dont fait partie a_j^{p-m}, et par conséquent de l'un des a_k^p dont fait partie a_i^q. Il suffit de le montrer pour les sommets de b_j^m ; or, si b_k^0 est l'un de ces sommets, il sera à l'intérieur de a_k^p, et comme b_k^0 appartient à b_j^m, en vertu de la définition même des polyèdres réciproques, a_j^{p-m} appartiendra à a_k^p.

Cela posé, nous pouvons à l'intérieur de chacun des a_k^p définir un système de lignes L, tel que par deux points quelconques intérieurs à cet a_k^p on puisse mener une ligne L, et une seule. Le système des lignes L jouit donc des mêmes propriétés qualitatives que le système des lignes droites. Cela tient à ce que a_k^p est supposé simplement connexe.

Joignons maintenant chacun des points de b_j^m à chacun des points de a_i^q par une ligne L située dans celui des a_k^p auquel appartient à la fois a_i^q et le point considéré de b_j^m.

L'ensemble de ces lignes L engendrera une figure que j'appellerai $a_i^q b_j^m$, qui sera homéomorphe à une pyramide généralisée rectiligne et qui aura $q + m + 1 \leqq p$ dimensions.

Quelle sera la frontière de cette variété $a_i^q b_j^m$? Supposons que l'on ait les congruences

$$a_i^q \equiv \Sigma \epsilon_{ih}^q a_h^{q-1}; \quad b_j^m \equiv \Sigma \epsilon_{jk}^{\prime m} b^{m-1}.$$

La frontière se composera des pyramides généralisées $a_h^{q-1} b_i^m$ et $a_i^q b_k^{m-1}$, et l'on aura

$$(7) \quad a_i^q b_j^m \equiv \Sigma \epsilon_{ih}^q a_h^{q-1} b_j^m + \Sigma \epsilon_{jk}^{\prime m} a_i^q b_k^{m-1}.$$

Cela ne serait plus vrai si l'on avait $m = 0$. Dans ce cas, en effet, la *variété* a_i^q aurait $q = (q + m + 1) - 1$ dimensions ; elle devrait donc faire partie de la frontière complète de $a_i^q b_j^m$, et la congruence (7) deviendrait

$$(7\text{ bis})\qquad a_i^q b_j^0 \equiv \Sigma \epsilon_{ih}^q a_h^{q-1} b_j^0 - a_i^q$$

(les termes en ϵ' disparaissant) ; de même pour $q = 0$ on aurait

$$(7\text{ ter})\qquad a_i^0 b_j^m \equiv \Sigma \epsilon_{jk}'^m a_i^0 b_k^{m-1} + b_j^m.$$

Des congruences (7), (7 bis) et (7 ter) se déduisent les homologies

$$(8)\qquad \Sigma \epsilon_{ih}^q a_h^{q-1} b_j^m \sim -\Sigma \epsilon_{jk}'^m a_i^q b_k^{m-1},$$

$$(8\text{ bis})\qquad a_i^q \sim \Sigma \epsilon_{ih}^q a_h^{q-1} b_j^0,$$

$$(8\text{ ter})\qquad b_j^m \sim -\Sigma \epsilon'^m_{jk} a_i^0 b_k^{m-1}.$$

La congruence (8 bis) suppose que a_i^q fasse partie de a_j^p ; c'est celle que nous avons envisagée ailleurs (c, § x., p. 49, éq. 2).

Supposons maintenant que nous ayons trouvé une congruence de la forme

$$(9)\qquad \Sigma \lambda_{ij} a_i^q b_j^m \equiv 0.$$

Je dis que nous pourrons trouver une congruence de la même forme, mais où le nombre q a augmenté d'une unité et le nombre m diminué d'une unité, et cela de telle sorte que les premiers membres des deux congruences soient homologues.

En effet, nous avons identiquement en vertu de (7)

$$\Sigma \lambda_{ij} a_i^q b_j^m \equiv \Sigma \lambda_{ij} \epsilon_{ih}^q a_h^{q-1} b_j^m + \Sigma \lambda_{ij} \epsilon_{jk}'^m a_i^q b_k^{m-1}.$$

On doit donc avoir (en annulant dans le second membre le coëfficient de $a_h^{q-1} b_j^m$)

$$\Sigma_i \lambda_{ij} \epsilon_{ih}^q = 0.$$

On en déduit la congruence

$$(10)\qquad \Sigma_i \lambda_{ij} a_i^q \equiv \Sigma_i \lambda_{ij} \epsilon_{ih}^q a_h^{q-1} \equiv 0.$$

Tous les éléments a_i^q qui figurent dans le premier membre de (10) appartiennent à a_j^{p-m} ; or a_j^{p-m} par hypothèse *est simplement connexe ;* toute congruence entre ses éléments entraîne donc l'homologie correspondante de sorte que l'on a

$$\Sigma_i \lambda_{ij} a_i^q \sim 0,$$

d'où

$$\Sigma_i \lambda_{ij} a_i^q \equiv \Sigma_\rho \mu_{\rho j} a_\rho^{q+1},$$

les μ étant des coëfficients entiers et les a_ρ^{q+1} étant des éléments appartenant à a_j^{p-m}.

Or

$$\Sigma_\rho \mu_{\rho j} a_\rho^{q+1} \equiv \Sigma_{\rho i} \mu_{\rho j} \epsilon_{\rho i}^{q+1} a_i^q.$$

On a donc

$$\lambda_i = \Sigma_\rho \mu_{\rho j} \epsilon_{\rho i}^{q+1}.$$

La congruence (9) peut alors s'écrire

$$\Sigma \mu_{\rho j} \epsilon_{\rho i}^{q+1} a_i^q b_j^m \equiv 0$$

(la sommation s'étend aux trois indices ρ, i, j).

Or nous pouvons former l'homologie suivante qui n'est autre que l'une des homologies (8) :—

$$(11) \quad \Sigma \epsilon_{\rho i}^{q+1} a_i^q b_j^m \sim -\Sigma \epsilon'^{m}_{jk} a_\rho^{q+1} b_k^{m-1}.$$

On a alors

$$\Sigma \lambda_{ij} a_i^q b_j^m \sim -\Sigma \mu_{\rho j} \epsilon'^{m}_{jk} a_\rho^{q+1} b_k^{m-1},$$

ce qui démontre le théorème énoncé.

Le cas de $m = 0$ est, bien entendu, laissé de côté et doit être traité à part. Dans ce cas l'homologie (11) doit être remplacée par la suivante qui est l'une des homologies (8 bis) :—

$$(11\ \text{bis}) \quad \Sigma \epsilon_{\rho i}^{q+1} a_i^q b_j^0 \sim a_\rho^{q+1},$$

d'où

$$\Sigma \lambda_{ij} a_i^q b_j^0 \sim \Sigma \mu_{\rho j} a_\rho^{q+1}.$$

Donc à la congruence

$$(9\ \text{bis}) \quad \Sigma \lambda_{ij} a_i^q b_j^0 \equiv 0$$

correspondra la congruence

$$\Sigma \mu_{\rho j} a_\rho^{q+1} \equiv 0$$

qui est de la forme (1), et les premiers membres de ces deux congruences seront homologues.

Soit maintenant

$$(2) \quad \Sigma \lambda_j b_j^q \equiv 0$$

une congruence de la forme (2) ; on aura par une homologie analogue à (8 ter)

$$b_j^q \sim -\Sigma \epsilon'^{q}_{jk} a_i^0 b_k^{q-1},$$

si b_i^{p-1} est l'un des éléments de P' auquel appartient b_i^q.

Nous avons donc l'homologie

$$\Sigma \lambda_j b_j^q \sim -\Sigma \lambda_j \epsilon'^{q}_{jk} a_i^0 b_k^{q-1},$$

de sorte qu'à notre congruence (2) correspondra une congruence

$$(12) \quad -\Sigma \lambda_j \epsilon'^{q}_{jk} a_i^0 b_k^{q-1} \equiv 0$$

dont le premier membre est homologue à celui de (2).

Si donc nous avons une congruence de la forme (2), nous en déduirons la congruence (12), qui est une congruence de la forme (9), où les nombres que nous appelions plus haut q et m ont respectivement *pour valeurs* 0 et $q-1$. Nous en déduirons ensuite une autre congruence également de la forme (9), mais où ces deux nombres auront pour valeurs 1 et $q-2$, et ainsi de suite ; on finira par arriver à une congruence de la forme (9 bis), c'est-à-dire à une congruence où ces deux nombres auront pour valeurs $q-1$ et 0 ; et nous en déduirons alors finalement une congruence de la forme (1).

Les premiers membres de toutes ces congruences seront homologues entre eux.

Le théorème énoncé au début de ce paragraphe se trouve ainsi démontré.

Pour en tirer toutes les conséquences qu'il comporte, nous devons remarquer ceci : Nous devons distinguer plusieurs sortes d'homologies. Soit v_q une variété quelconque à q dimensions faisant partie de notre variété v, et v_{q-1} sa frontière complète, ce qui s'exprime par la congruence

$$v_q \equiv v_{q-1}.$$

Nous en déduisons l'homologie

$$v_{q-1} \sim 0.$$

Les homologies ainsi obtenues sont les homologies fondamentales.

En combinant les homologies fondamentales par addition, soustraction et multiplication, on en obtient d'autres qui sont les *homologies sans division.* Enfin, en les combinant par addition, multiplication et division, on en obtient encore d'autres qui sont les *homologies par division.*

Eh bien, *toutes les homologies que nous avons rencontrées dans ce paragraphe sont des homologies sans division.*

Cela posé, revenons à nos tableaux T_q et T'_q et à leurs invariants, et en particulier à ceux de ces invariants qui ne sont égaux ni à 0, ni à 1, et que nous appellerons *coëfficients de torsion.*

Supposons que nous ayons l'homologie suivante :—

$$(13) \quad \Sigma k \lambda_i a_i^q \sim 0,$$

où les λ_i sont des entiers premiers entre eux ; que (13) soit une homologie sans division, mais que l'homologie

$$(14) \quad \Sigma \lambda_i a_i^q \sim 0$$

ne puisse être obtenue que par division. D'après ce que nous avons vu dans l'un des paragraphes précédents, cela voudra dire que k est l'un des coëfficients de torsion du tableau T_q.

Nous aurons la congruence

$$(14\text{ bis}) \quad \Sigma \lambda_i a_i^q \equiv 0.$$

De (14 bis) nous pourrons, par le procédé de ce paragraphe, déduire une congruence entre les b_i^q que j'écrirai

$$(14\text{ ter})\quad \Sigma\mu_i b_i^q \equiv 0.$$

On aurait d'ailleurs, d'après le théorème que nous venons d'établir,

$$\Sigma\lambda_i a_i^q \sim \Sigma\mu_i b_i^q.$$

C'est là une homologie sans division, et on en déduirait immédiatement, également sans division,

$$\Sigma k\lambda_i a_i^q \sim \Sigma k\mu_i b_i^q.$$

De là on déduit que l'on a, sans division,

$$\Sigma k\mu_i b_i^q \sim 0,$$

et que l'on n'a pas, sans division,

$$\Sigma\mu_i b_i^q \sim 0,$$

sans quoi l'on aurait, sans division,

$$\Sigma\lambda_i a_i^q \sim 0,$$

ce qui est contraire à l'hypothèse.

Cela veut dire que k est un coëfficient de torsion du tableau T'_q.

Ainsi les coëfficients de torsion des deux tableaux T_q et T'_q sont égaux (la démonstration est aisée à compléter), et, si l'on observe que les deux tableaux T'_q et T_{p-q} ont mêmes invariants, on conclura que *les tableaux également distants des extrêmes ont mêmes coëfficients de torsion.*

On pourrait arriver au même résultat par une autre voie.

Nous avons vu dans un des mémoires antérieurs (§ 16) définir l'opération que nous avons appelée l'annexion; je suppose que deux éléments d'un polyèdre, par exemple a_i^q et a_j^q, soient séparés l'un de l'autre par un élément a_k^{q-1}, que ce soit le seul élément à $q-1$ dimensions commun à a_i^q et à a_j^q, et enfin que a_k^{q-1} n'appartienne à aucun élément à q dimensions en dehors de a_i^q et de a_j^q; on aura donc $\epsilon_{ik}^q = 1$, $\epsilon_{jk}^q = -1$; tous les autres ϵ_{hk}^q seront nuls quel que soit l'indice h, de même que tous les produits $\epsilon_{ih}^q \epsilon_{jh}^q$.

Dans ces conditions, on peut annexer l'un à l'autre les deux éléments a_i^q et a_j^q en supprimant l'élément a_k^{q-1}. Quel est l'effet de cette opération sur nos tableaux T_q? Le tableau T_q perd une ligne et une colonne; le tableau T_{q-1} perd une ligne. L'un des invariants égaux à *1 de T_q disparaît*; quant au tableau T_{q-1}, il perd un invariant s'il n'a pas plus de lignes que de colonnes ; dans ce cas, l'invariant qu'il perd est égal à zéro. Tous les autres invariants des deux tableaux ne changent pas ; ces deux tableaux conservent donc leurs coëfficients de torsion.

Or il est aisé de former un polyèdre dérivé à la fois de P et de P' ; on pourrait ensuite remonter de ce polyèdre soit à P, soit à P', par des annexions régulières. Comme ces annexions n'altèrent pas les coëfficients de torsion, les tableaux T_q et T'_q doivent avoir mêmes coëfficients de torsion.

6. *Torsion intérieure des Variétés.*

Considérons l'un de nos tableaux T_q. Nous dirons qu'une suite d'éléments, tous distincts, de ce tableau, rangés dans un certain ordre forme une *chaîne*, si chaque élément de rang impair appartient à la même ligne que l'élément suivant et à la même colonne que l'élément précédent. La chaîne sera *fermée* si le dernier élément est identique au premier. Il est clair qu'une chaîne fermée contiendra toujours un nombre impair d'éléments et un nombre pair d'éléments ***distincts.*** Par exemple, les éléments

$$(1) \quad \epsilon^q_{11},\ \epsilon^q_{12},\ \epsilon^q_{22},\ \epsilon^q_{23},\ \epsilon^q_{33},\ \epsilon^q_{31},\ \epsilon^q_{11}$$

formeront une chaîne fermée.

Comme tous les éléments du tableau T_q sont égaux à 0, +1 ou −1, le produit des éléments distincts d'une chaîne fermée sera toujours 0, +1 ou −1.

Supposons que les éléments de la chaîne (1) aient les valeurs suivantes :—

$$\epsilon^q_{12} = \epsilon^q_{23} = \epsilon^q_{31} = 1, \quad \epsilon^q_{11} = \epsilon^q_{22} = \epsilon^q_{33} = -1 \ ;$$

le produit des éléments de la chaîne sera −1 ; considérons alors les trois variétés a^q_1, a^q_2, a^q_3, et les trois variétés a^{q-1}_1, a^{q-1}_2, a^{q-1}_3 ; en supprimant les variétés a^{q-1}_1, a^{q-1}_2 et a^{q-1}_3, on annexe les unes aux autres les trois variétés a^q_1, a^q_2 et a^q_3, et la variété ainsi obtenue

$$a^q_1 + a^q_2 + a^q_3$$

est une variété bilatère.

Si, au contraire, nous avons

$$\epsilon^q_{12} = \epsilon^q_{23} = \epsilon^q_{31} = 1, \quad \epsilon^q_{22} = \epsilon^q_{33} = -1, \quad \epsilon^q_{11} = 1,$$

on pourra encore supprimer a^{q-1}_1, a^{q-1}_2 et a^{q-1}_3 et obtenir par annexion la variété $a^q_1 + a^q_2 + a^q_3$; mais *cette variété sera unilatère.*

Plus généralement, si tous les éléments de la chaîne (1) sont égaux à $+1$ et à -1, nous supprimerons d'abord a_2^{q-1} et a_3^{q-1} ; nous obtiendrons ainsi par annexion la variété

$$(2) \quad a_1^q - \epsilon_{12}^q \epsilon_{22}^q a_2^q + \epsilon_{12}^q \epsilon_{22}^q \epsilon_{13}^q \epsilon_{23}^q a_3^q.$$

Supprimant ensuite a_1^{q-1}, nous voyons que la variété (2) est désormais formée d'une chaîne fermée de a_i^q au sens du paragraphe 8 (p. 26) de l'"Analysis Situs," et que cette chaîne est bilatère ou unilatère selon que le produit des éléments distincts de la chaîne (1) est égal à -1 ou à $+1$.

Nous dirons dans le premier cas que la chaine (1) est bilatère, dans le second cas qu'elle est unilatère.

Nous sommes donc conduits à distinguer trois catégories parmi les chaînes fermées formées à l'aide d'éléments des tableaux T_q :

1° Les chaînes *nulles*, c'est-à-dire celles dont le produit des éléments est nul.

2° Les chaînes *bilatères*.

Il est aisé de voir que ce sont celles dont le produit des éléments est $+1$ si le nombre des éléments est multiple de 4, ou celles où ce produit est -1 si le nombre des éléments est multiple de 4 plus 2.

3° Les chaînes *unilatères*.

Ce sont celles où ce produit est -1 si le nombre des éléments est multiple de 4, ou $+1$ si ce nombre est multiple de 4 plus 2.

Cela posé, nous dirons que le tableau T_q (ou plus généralement tout tableau ou tout déterminant dont tous les éléments sont 0, $+1$ ou -1) est *bilatère* s'il ne contient que des chaînes nulles ou bilatères.

Il résulte de cette définition :

1° Qu'un tableau bilatère reste bilatère si l'on change tous les signes d'une colonne, ou tous les signes d'une ligne ; ou encore si l'on permute deux colonnes ou deux lignes.

Théorème.—Un déterminant bilatère ne peut être égal qu'à 0, $+1$ ou -1.

En effet, on peut toujours, en changeant au besoin tous les signes de certaines colonnes, s'arranger de façon que tous les éléments de la première ligne soient 0 ou $+1$.

Supposons, par exemple, que les deux premiers éléments de la première ligne soient égaux à $+1$, et que je retranche la première colonne de la seconde, la valeur du déterminant ne sera pas changée ; je dis que le déterminant restera bilatère.

Considérons, en effet, dans le déterminant primitif une chaîne dont *le premier et le* dernier élément appartiennent à la deuxième colonne et tous les autres éléments à d'autres colonnes. Soient a et c ce premier et ce dernier élément; soit ξ le produit de tous les autres éléments de la chaîne; soient b et d les éléments de la première colonne qui sont respectivement dans la même ligne que a et c.

Le produit des éléments de notre chaîne que j'appellerai la chaîne (1) sera $ac\xi$, et nous aurons

$$ac\xi = 0 \text{ ou } \;\; 1 \;\; \text{si le nombre des éléments} \equiv 0 \pmod 4,$$

$$ac\xi = 0 \text{ ou } -1 \quad ,, \qquad ,, \qquad ,, \quad \equiv 2 \pmod 4.$$

Le produit des éléments de la chaîne que j'appellerai (2) et qui est formée avec les éléments correspondants du déterminant nouveau sera

$$(a-b)\,(c-d)\,\xi,$$

et, en effet, les éléments de notre chaîne ne changent pas, excepté les éléments a et c qui deviennent $a-b$ et $c-d$.

La chaîne formée dans le déterminant primitif par les deux éléments de la première ligne et par les éléments a et b doit être bilatère, de sorte qu'on doit avoir

$$a-b=0 \quad \text{ou} \quad a=0 \quad \text{ou} \quad b=0.$$

On doit avoir de même

$$c-d=0 \quad \text{ou} \quad c=0 \quad \text{ou} \quad d=0.$$

Si $(a-b)$ ou $(c-d)$ est nul, le théorème est démontré puisque le produit $(a-b)\,(c-d)\,\xi=0$.

Si $b=d=0$, on a

$$(a-b)\,(c-d)\,\xi = ac\xi,$$

et le théorème est démontré puisque les deux produits des chaînes (1) et (2) sont les mêmes, que le nombre des éléments est le même et que (1) est bilatère ou nulle.

Si $a=c=0$, on a

$$(a-b)\,(c-d)\,\xi = bd\xi.$$

La chaîne (3) qui appartient au déterminant primitif, et qui a mêmes éléments que la chaîne (1), sauf que a et c sont remplacés par b et d—cette chaîne (3), dis-je, est bilatère ou nulle; elle a même nombre d'éléments que (2) et son produit est $bd\xi$, égal dans ce cas au produit de (2). Donc, dans ce cas encore, la chaîne (2) est bilatère ou nulle.

Si $a=d=0$, on a

$$(a-b)\,(c-d)\,\xi = -bc\xi.$$

Il faut cette fois considérer dans le déterminant primitif une chaîne (4) dont les éléments seront les deux éléments de la première ligne, les éléments b et c et les éléments de la chaîne (1), sauf a et c. Cette chaîne (4) doit être bilatère ou nulle.

Elle contient deux éléments de plus que la chaîne (2).

Son produit est égal à $bc\xi$ et, par conséquent, égal et de signe contraire au produit de (2).

Donc (2) est bilatère ou nulle.

Si, enfin, $b = c = 0$, on a

$$(a-b)(c-d)\xi = -ad\xi,$$

et on démontrerait, tout à fait comme dans le cas précédent, que la chaîne (2) est bilatère ou nulle.

Nous venons de traiter le cas des chaînes dont deux éléments appartiennent à la seconde colonne. Le résultat est le même quel que soit le nombre des éléments appartenant à la seconde colonne, nombre qui d'ailleurs doit être toujours pair.

Si ce nombre est nul, le théorème est évident, car la chaîne du déterminant nouveau ne diffère pas de celle du déterminant primitif.

Supposons que ce nombre soit 4, pour fixer les idées. Soient a, c, e, g quatre éléments de la seconde colonne, et imaginons que l'on rencontre successivement l'élément a, divers éléments ξ appartenant à d'autres colonnes, les éléments c et e, divers éléments η appartenant à d'autres colonnes, et enfin g. Notre chaîne sera fermée.

$$a\xi ce\eta ge$$

peut se décomposer en deux chaînes fermées

$$a\xi ca, \quad e\eta ge,$$

et, pour qu'elle soit bilatère, il suffit que les deux composantes le soient. On est donc ramené au cas des chaînes n'ayant que deux éléments dans la seconde colonne.

J'ajouterai que tous les éléments du déterminant nouveau sont 0. +1 ou −1. En effet, comme on a

$$a-b=0 \quad \text{ou} \quad a=0 \quad \text{ou} \quad b=0,$$

on aura

$$a-b=0, \quad a \quad \text{ou} \quad -b,$$

d'où

$$a-b=0, \quad 1 \quad \text{ou} \quad -1.$$

Cela posé, retranchons de cette façon la première colonne de toutes les colonnes dont le premier élément est +1. Le déterminant conservera sa valeur; il restera bilatère; mais tous les éléments de la *première ligne* seront nuls, sauf le premier qui sera +1.

Ce raisonnement est applicable dans tous les cas, sauf si tous les éléments de la première ligne sont nuls ; mais alors le déterminant est nul et le théorème est évident.

Si maintenant on supprime la première ligne et la première colonne, on obtiendra un déterminant nouveau qui sera égal au premier et, comme lui, bilatère. Sur ce déterminant nouveau, qui a une ligne et une colonne de moins que le premier, on opérera de la même façon, et on finira par arriver à un déterminant qui n'aura plus qu'un seul élément, lequel devra être 0, $+1$ ou -1.

Notre déterminant est donc égal à 0, $+1$ ou -1.

1^er^ *Corollaire.*—Si un tableau T_q est bilatère, ses invariants sont tous 0 ou 1.

2^e^ *Corollaire.*—Si un polyèdre a tous ses tableaux T_q bilatères, c'est-à-dire si on ne peut pas composer avec ses éléments a_i^q une variété unilatère, ce polyèdre n'a pas de coëfficients de torsion.

On voit que l'existence des coëfficients de torsion (qui nécessite la distinction entre les deux définitions des nombres de Betti, ou entre les homologies par division ou sans division) est due à ce fait que les éléments du polyèdre peuvent engendrer des variétés unilatères, c'est-à-dire que le polyèdre est pour ainsi dire tordu sur lui-même.

C'est ce qui justifie l'expression de coëfficients de torsion, ou celle de variétés avec ou sans torsion.

Si la variété V formée par l'ensemble des éléments a_i^p du polyèdre P n'est pas elle-même unilatère, les deux tableaux T_1 et T_p sont bilatères.

En effet, chaque ligne pour l'un, chaque colonne pour l'autre a tous ses éléments nuls, sauf deux qui sont égaux à $+1$ et -1. Si donc une chaîne n'est pas nulle, ses éléments sont deux à deux égaux et de signe contraire ; elle est donc bilatère.

Il résulte de là que les tableaux extrêmes T_1 et T_p ont tous leurs invariants égaux à 0 ou à 1. C'est ce qui explique pourquoi l'on ne rencontre pas les coëfficients de torsion avec les polyèdres de l'espace ordinaire ; ces polyèdres ne comportent en effet que deux tableaux T_1 et T_2.

Cela ne serait plus vrai si la variété V était unilatère. Ainsi la variété considérée au septième exemple (§ 15, p. 87) peut être subdivisée en polyèdre, et, suivant la manière dont la subdivision se fait, on trouve pour le tableau T_2

$$|2|, \quad \begin{vmatrix} +1 & +1 \\ +1 & -1 \end{vmatrix}, \quad \ldots\ldots\ldots$$

x 2

Pour ne pas trop allonger ce travail, je me bornerai à énoncer le théorème suivant dont la démonstration demanderait quelques développements :—

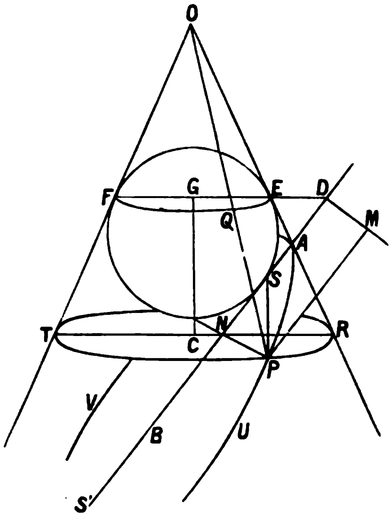
Tout polyèdre qui a tous ses nombres de Betti égaux à **1** *et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire homéomorphe à l'hypersphère.*

A Proof of the Directro-Focal Property of the Plane Sections of a Cone in non-Euclidean Space. By Irving Stringham, Ph.D.

Communicated June 14th, 1900. Received June 18th, 1900.

In the analysis of the homogeneous equation of the second degree in three variables, interpreted as coordinates in the non-Euclidean plane, it is made evident that the curve represented by that equation possesses the directro-focal property, but from this fact it does not follow that the curve exists also as the plane section of a cone. In order to make sure that the curve is really a conic section, it must be shown geometrically that the plane sections of a cone have also the directro-focal property. I have long supposed that such a geometrical proof exists somewhere, and I still suspect that Prof. Killing has one in his possession, but my search for it, in the literature of the subject, has thus far resulted unsuccessfully. The following proof supplies the missing link and is offered as a possible contribution to the subject. It should be noted that the argument is applicable in either elliptic or hyperbolic geometry.

The right circular cone is here defined, in the usual manner, as the surface generated by a straight line turning about a fixed point and forming with a fixed straight line through this point a constant angle. Any plane meets the cone in a conic section AUV. A sphere FES may be so constructed as to touch the cone in the circle FQE (its centre at G) and the plane in the point S. The plane of the circle is perpendicular to the axis of the cone, GC. Through any point on the conic section, as P, pass a plane, also perpendicular to the axis of the cone, forming with the conical surface the circle TPR, whose centre is at *C*. These two planes meet the plane of the conic section in two straight lines, *DM*, *NP*, which are met at right angles



in *D* and *N* by the principal diameter, *AB*, of the conic section. *PS*, *PQ*, and *RE* are tangent to the sphere, and are therefore equal to one another in length, and *PM* is drawn perpendicular to *DM*.

In the figure thus drawn there are two birectangular quadrilaterals *NCGD* and *RCGE*, with right angles at *C* and *G*, and *PNDM* is a trirectangular quadrilateral with its right angles at *N*, *D*, and *M*. The following relations between the parts of these figures are well known.* In *NCGD*,

$$\sin_\kappa ND = \frac{\sin_\kappa CG \cos_\kappa CN}{\sin \delta},$$

where δ stands for the angle *GDN*; and, in *RCGE*,

$$\sin_\kappa RE = \frac{\sin_\kappa CG \cos_\kappa CR}{\sin \epsilon},$$

where ϵ stands for the angle *GER*; and, in *PMDN*,

$$\sin_\kappa PM = \sin_\kappa ND \cos_\kappa NP;$$

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* *Cf.* F. Hausdorff, "Analytische Beiträge zur nichteuklidischen Geometrie," in *Leipziger Berichte*, Math.-Phys. Classe, Math. Theil, 1899, p. 163.

and, finally, in the triangle *CNP*,

$$\cos_\kappa CN \cos_\kappa NP = \cos_\kappa CP = \cos_\kappa CR,$$

the functions $\sin_\kappa$ and $\cos_\kappa$ being here defined by the identities

$$\sin_\kappa u \equiv \frac{\kappa}{2}\left(e^{u/\kappa} - e^{-u/\kappa}\right),$$

$$\cos_\kappa u \equiv \tfrac{1}{2}\left(e^{u/\kappa} + e^{-u/\kappa}\right),$$

where κ is the fundamental constant, or modulus, of the space-form in question, and is in the general case a complex number.

It follows immediately that

$$\frac{\sin_\kappa PS}{\sin_\kappa PM} = \frac{\sin \delta}{\sin \epsilon} = e, \text{ a constant.}$$

This expresses the directro-focal property of the conic section. *S* is a focus, *DM* is the directrix, and *e* is the eccentricity.

This ratio has precisely the form called for by the properly transformed homogeneous equation of the second degree. If $\bar{r}$ and θ be the radius vector and vectorial angle of the point (w, x, y), and $\bar{p}$, ν be the normal and norm-angle to the straight line (ω, α, β), and if as definitions of w, x, y and ω, α, β we adopt the system of values

$$w, x, y = i\kappa \cos_\kappa \bar{r}, \;\; \sin_\kappa \bar{r} \sin\theta, \;\; \sin_\kappa \bar{r} \cos\theta,$$

$$\omega, \alpha, \beta = -\frac{\sin_\kappa \bar{p}}{i\kappa}, \;\; \cos_\kappa p \sin\nu, \;\; \cos_\kappa \bar{r} \cos\nu,$$

the distance formulæ for *PS* and *PM*, in the notation here employed, are, for *PS*,

$$(\sin_\kappa PS)^2 = \kappa^{-2}\left([yw']^2 + [wx']^2 + [xy']^2\right),$$

where $[yw']$, $[wx']$, and $[xy']$ are abbreviations for the determinants $yw' - y'w$, $wx' - w'x$, &c., and, for *PM*,

$$(\sin_\kappa PM)^2 = (\omega w + \alpha x + \beta y)^2.$$

Hence the directro-focal property is expressed by the equation

$$[yw']^2 + [wx']^2 + [xy']^2 + \kappa^2 e^2 (\omega w + \alpha x + \beta y)^2 = 0,$$

and this is homogeneous and of the second degree in w, x, y.

Of its seven constants (κ must be treated as given) five, namely, x', y', α, β, and e, may be regarded as independent, the other two being given by

$$w' = \sqrt{(-x'^2 - y'^2 - \kappa^2)},$$

$$\omega = \sqrt{(1 - \alpha^2 - \beta^2)},$$

the two fundamental equations which may be assigned as the conditions that w, x, y, and ω, α, β shall be the coordinates of a point and of a straight line respectively. It will therefore be possible to write these constants as functions of the five independent coefficients of the general homogeneous equation of the second degree in w, x, y, and thereby reduce this equation to the above special form, which expresses directly the directro-focal property. Hence:

The general homogeneous equation of the second degree in three variables, interpreted as point coordinates, represents a conic section.

The Energy Function of a Continuous Medium transmitting Transverse Waves.

By H. M. MACDONALD.

Read and received June 14th, 1900. Received, in revised form, August 8th, 1900.

The object of the following note is to obtain the most general form of the energy function of a homogeneous continuous medium, assumed to be a homogeneous quadratic function of the nine first differential coefficients of the displacements which satisfies Green's condition that transverse waves are propagated in it independently of waves of dilatation. It is further shown that when this condition is satisfied the wave surface is Fresnel's.

Denoting the energy function by ϕ, then

$$\begin{aligned} 2\phi = a_{11}e^2 + a_{22}f^2 + a_{33}g^2 + a_{44}a^2 + a_{55}b^2 + a_{66}c^2 \\ + a_{77}\omega_1^2 + a_{88}\omega_2^2 + a_{99}\omega_3^2 + a_{12}ef + a_{14}ea \\ + a_{45}ab + a_{17}e\omega_1 + a_{47}a\omega_1 + a_{78}\omega_1\omega_2 + \ldots, \end{aligned}$$

where e, f, g, a, b, c are the component extensions and shears, and ω_1, ω_2, ω_3 are double the component rotations.

The equations of motion are

$$\rho\ddot{u} = \frac{\partial}{\partial x}\phi_{u_1} + \frac{\partial}{\partial y}\phi_{u_2} + \frac{\partial}{\partial z}\phi_{u_3},$$

&c.,

where u_1, u_2, u_3 denote $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, &c., and ϕ_{u_1} denotes the differential coefficient of ϕ with respect to u_1, &c. These are equivalent to

$$\begin{aligned}\rho\ddot{u} = {} & \frac{\partial}{\partial x}\left[a_{11}e+a_{12}f+a_{13}g+a_{14}a+a_{15}b+a_{16}c+a_{17}\omega_1+a_{18}\omega_2+a_{19}\omega_3\right] \\ & +\frac{\partial}{\partial y}\left[(a_{61}-a_{91})e+(a_{62}-a_{92})f+(a_{63}-a_{93})g+(a_{64}-a_{94})a\right. \\ & \qquad +(a_{65}-a_{95})b+(a_{66}-a_{96})c+(a_{67}-a_{97})\omega_1+(a_{68}-a_{98})\omega_2 \\ & \qquad \left.+(a_{69}-a_{99})\omega_3\right] \\ & +\frac{\partial}{\partial z}\left[(a_{51}+a_{81})e+(a_{52}+a_{82})f+(a_{53}+a_{83})g+(a_{54}+a_{84})a\right. \\ & \qquad +(a_{55}+a_{85})b+(a_{56}+a_{86})c+(a_{57}+a_{87})\omega_1+(a_{58}+a_{88})\omega_2 \\ & \qquad \left.+(a_{59}+a_{89})\omega_3\right],\end{aligned}$$

with two similar equations. For plane waves assume

$$u = Ae^{\iota\kappa(lx+my+nz+Vt)},$$

$$v = Be^{\iota\kappa(lx+my+nz+Vt)},$$

$$w = Ce^{\iota\kappa(lx+my+nz+Vt)};$$

then, if this represents a wave of dilatation which is propagated independently of transverse waves,

$$A/l = B/m = C/n,$$

and substituting in the equations of motion

$$\begin{aligned}l\rho V^2 = {} & a_{11}l^3+a_{12}lm^2+a_{13}ln^2+2a_{14}lmn+2a_{15}l^2n \\ & +2a_{16}l^2m+(a_{61}-a_{91})l^2m+(a_{62}-a_{92})m^3+(a_{63}-a_{93})mn^2 \\ & +2(a_{64}-a_{94})m^2n+2(a_{65}-a_{95})lmn+2(a_{66}-a_{96})lm^2 \\ & +(a_{51}+a_{81})l^2n+(a_{52}+a_{82})m^2n+(a_{53}+a_{83})n^3 \\ & +2(a_{54}+a_{84})mn^2+2(a_{55}+a_{85})ln^2+2(a_{56}+a_{86})lmn,\end{aligned}$$

with two similar equations, true for all values of l, m, n. Hence, multiplying the left-hand side by $l^2+m^2+n^2$, which is unity,

$$a_{11} = a_{12}+2a_{66}-2a_{96} = a_{13}+2a_{55}+2a_{85} = \rho V^2,$$

$$a_{14}+2a_{56}-a_{95}+a_{86} = 0,$$

$$3a_{15}+a_{81} = 0,$$

$$3a_{16}-a_{91} = 0,$$

$$a_{62}-a_{92} = 0,$$

$$a_{63}-a_{93}+2\,(a_{54}+a_{84}) = 0,$$

$$a_{52}+a_{82}+2\,(a_{64}-a_{94}) = 0,$$

$$a_{53}+a_{83} = 0,$$

$$a_{22} = a_{23}+2a_{44}-2a_{74} = a_{12}+2a_{66}+2a_{96} = \rho V^2,$$

$$a_{25}+2a_{46}+a_{94}-a_{76} = 0,$$

$$a_{61}+a_{91} = 0,$$

$$3a_{63}+a_{92} = 0,$$

$$3a_{42}-a_{72} = 0,$$

$$a_{63}+\ a_{93}+2a_{45}-2a_{75} = 0,$$

$$2a_{65}+2a_{95}+\ a_{41}-\ a_{71} = 0,$$

$$a_{43}-a_{73} = 0,$$

with a similar group. From these it follows that

$$a_{11} = a_{22} = a_{33} = \rho V^2,$$

$$a_{23}+2a_{44} = a_{31}+2a_{55} = a_{12}+2a_{66} = \rho V^2,$$

$$a_{74} = a_{85} = a_{96} = 0,$$

$$a_{91} = a_{61} = 0, \qquad a_{92} = a_{62} = 0,$$

$$a_{81} = a_{51} = 0, \qquad a_{83} = a_{53} = 0,$$

$$a_{72} = a_{42} = 0, \qquad a_{73} = a_{43} = 0,$$

$$a_{63}+2a_{45} = 0, \qquad a_{25}+2a_{64} = 0, \qquad a_{14}+2a_{65} = 0,$$

$$a_{17} = 2a_{86} = 2a_{95},$$

$$a_{28} = 2a_{67} = 2a_{94},$$

$$a_{39} = 2a_{57} = 2a_{84};$$

and therefore the energy function is given by

$$\begin{aligned}2\phi = K\,(e+f+g)^2 &+ a_{44}\,(a^2-4fg)+a_{55}\,(b^2-4ge)\\ &+a_{66}\,(c^2-4ef)-a_{14}\,(bc-2ea)-a_{25}\,(ca-2fb)\\ &-a_{36}\,(ab-2gc)+a_{17}\,(2e\omega_1+c\omega_2+b\omega_3)\\ &+a_{28}\,(c\omega_1+2f\omega_2+a\omega_3)+a_{39}\,(b\omega_1+a\omega_2+2g\omega_3)\\ &+(c_{11},\ c_{22},\ c_{33},\ c_{12},\ c_{13},\ c_{23})(\omega_1,\ \omega_2,\ \omega_3)^2,\end{aligned}$$

where $$K = a_{11}.$$

Now $$2e\omega_1+c\omega_2+b\omega_3 = 2\left(\omega_1\frac{\partial u}{\partial x}+\omega_2\frac{\partial u}{\partial y}+\omega_3\frac{\partial u}{\partial z}\right),$$

$$a^2-4fg = \omega_1^2+4\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right),$$

$$bc-2ea = -\omega_2\omega_3+2\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}\right).$$

&c.; hence $$2\iiint\phi\,dx\,dy\,dz$$

$$\begin{aligned}=\iiint\{K(e+f+g)^2&+(a_{44}+c_{11})\,\omega_1^2+(a_{55}+c_{22})\,\omega_2^2+(a_{66}+c_{33})\,\omega_3^2\\ &+2\,(a_{14}+c_{23})\,\omega_2\omega_3+2\,(a_{25}+c_{31})\,\omega_3\omega_1\\ &+2\,(a_{36}+c_{12})\,\omega_1\omega_2\}\,dx\,dy\,dz\\ &+a_{17}\iint u\,(l\omega_1+m\omega_2+n\omega_3)\,dS+\&c.\\ &+a_{44}\iint 4w\left(n\frac{\partial v}{\partial z}-m\frac{\partial v}{\partial y}\right)dS+\&c.\\ &-a_{74}\iint 2u\left(m\frac{\partial w}{\partial x}+n\frac{\partial v}{\partial x}-l\frac{\partial w}{\partial y}-l\frac{\partial v}{\partial z}\right)dS-\&c.\end{aligned}$$

Transforming to the principal axes of the quadrics

$$\begin{aligned}(a_{41}+c_{11})\,x^2+(a_{55}+c_{22})\,y^2+(a_{66}+c_{33})\,z^2&\\ +2\,(a_{14}+c_{23})\,yz+2\,(a_{25}+c_{31})\,zx+2\,(a_{36}+c_{12})\,xy &= \text{const.},\end{aligned}$$

the above takes the form

$$2\iiint\phi\,dx\,dy\,dz = \iiint\{K(e+f+g)^2+C_1\omega_1^2+C_2\omega_2^2+C_3\omega_3^2\}\,dx\,dy\,dz$$
$$+\text{ a surface integral};$$

and for this it is known that the wave surface for transverse waves is Fresnel's. Thus, whenever Green's condition, that transverse waves can be propagated independently of waves of dilatation in a continuous medium whose energy function is a homogeneous quadratic function of the nine first differential coefficients of the displacements, is satisfied, the wave-surface is Fresnel's.

In Green's * theory, in which the medium is supposed subjected to the influence of extraneous forces, the energy function due to the

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* *Mathematical Papers*, p. 298.

displacements of the medium from its state of rest under the action of no forces to its state of rest under the action of these extraneous forces is $\frac{1}{2}(As_1+Bs_2+Cs_3)$, where s_1, s_2, s_3 are the principal elongations, and Green's expression for the energy function for small motions is given by

$$\begin{aligned}2\phi = {} & 2\left(A\frac{\partial u}{\partial x}+B\frac{\partial v}{\partial y}+C\frac{\partial w}{\partial z}\right)\\ & +A\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right\}\\ & +B\left\{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}\\ & +C\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}\\ & +G\left(\frac{\partial u}{\partial x}\right)^2+H\left(\frac{\partial v}{\partial y}\right)^2+I\left(\frac{\partial w}{\partial z}\right)^2\\ & +2P\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}+2Q\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}+2R\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\\ & +L\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+M\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+N\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2.\end{aligned}$$

It follows from the above that, if the condition that transverse waves are propagated independently of waves of dilatation is satisfied, the extraneous forces must be such that $A = B = C$; the wave surface is then Fresnel's, and the direction of vibration is related to the plane of polarization in the same way as when there are no extraneous forces. In Green's* investigation, the condition of perfect transversality is violated, except in the case of the three principal waves.

The above investigation applies to any continuous medium which is isotropic as regards inertia, provided it is homogeneous as regards elastic properties. If the medium is æolotropic as regards inertia, a similar investigation applies, which shows that when Green's condition of perfect transversality is satisfied the wave surface is a surface derived from Fresnel's wave surface by a homogeneous strain.

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* *Loc. cit.*, p. 305.

Note on the Representation of a Circle by a Linear Equation. *By* John Griffiths, M.A.

Received and communicated June 14th, 1900.

[The coordinates x, y, z of the present note have been employed by me to prove various theorems in the geometry of the circle and triangle. See *Proc. Lond. Math. Soc.*, Vols. xxv., xxvi., and other vols. Hence it seems desirable to give a simpler interpretation of them than I have hitherto done.]

1. In the ordinary Cartesian method of determining the position of a point P in a plane, the equation $x = h$ represents a right line, every point on which has the same abscissa h. In like manner, it may be possible to choose a coordinate x, so that $x = h$ shall represent a circle or curve. In other words, the position of P may possibly be fixed by considering it as the intersection not only of straight lines, but also of circles, or curves. For example, suppose a circle to be drawn through the vertices B, C of a triangle ABC to intersect the sides AB, AC in X, X'. See Fig. 1. This curve will be completely determined when either of the abscissæ AX, AX' is known. Hence we may take AX or AX', or a function of them, to be the coordinate of a point P on a circle through B, C.

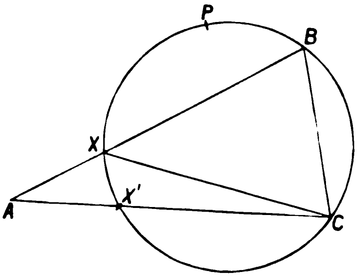

Fig. 1. $x = \dfrac{b}{AX} = \dfrac{\sin\theta}{\sin(\theta + A)}$, if $\angle BXC = \pi - \theta$.

In particular, if we denote either of the equal ratios $\dfrac{AC}{AX}$, $\dfrac{AB}{AX'}$ by x, there is no difficulty in expressing x in terms of the trilinear coordinates α, β, γ of P.

Here the function $\alpha . \Sigma a\alpha \div \Sigma a\beta\gamma$ is an invariant. Also, since x is an invariant for every point on the circle, we can assume

$$f(\alpha, \beta, \gamma) = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma = F(x).$$

This gives $$f(\alpha, \beta, 0) = f(\alpha, 0, \gamma) = F(x),$$

i.e., $$\frac{bc \sin A}{c . AX \sin A} = \frac{bc \sin A}{b . AX' \sin A} = F(x),$$

or $$x = \frac{b}{AX} = \frac{c}{AX'} = F(x) = f(\alpha, \beta, \gamma) = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma.$$

We thus have one coordinate, viz.,

$$x = \frac{b}{AX} = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

for fixing the position of a point $P\ (\alpha, \beta, \gamma)$.

Similarly, if two circles CPA, APB intersect the sides BC, CA, respectively, in Y and Z, we obtain two other coordinates

$$y = c \div BY = \beta . \Sigma a\alpha \div \Sigma a\beta\gamma;$$

$$z = a \div CZ = \gamma . \Sigma a\alpha \div \Sigma a\beta\gamma.$$

The position of P will then, in general, be uniquely determined by given values of the coordinates x, y, z.

It is, of course, obvious that there must be the fundamental relation $\Sigma ax = \Sigma ayz$ between x, y, z, since three arbitrary circles through BC, CA, AB will not necessarily meet in a common point.

2. Taking x, y, z to mean the three coordinates defined above, viz.,

$$x = b \div AX = \alpha . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

$$y = c \div BY = \beta . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

$$z = a \div CZ = \gamma . \Sigma a\alpha \div \Sigma a\beta\gamma,$$

it follows that a linear non-homogeneous relation $lx + my + nz = k$ will represent a circle, since the equivalent trilinear form is

$$(l\alpha + m\beta + n\gamma)\, \Sigma a\alpha = k . \Sigma a\beta\gamma.$$

[It is worth noticing here that the relations $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$, between the tricircular and trilinear coordinates of P, can be readily deduced by elementary geometry.]

3. As x, y, z may have any values between $-\infty$ and $+\infty$, it is important to consider what convention is to be adopted with regard

to the signs of the intercepts or abscissæ AX, BY, CZ. Now, since $AX = b \div x$, it is clear that AX is positive or negative according as x is positive or negative. Hence measure off AX in the direction AB or BA according as x is positive or negative.

Similarly, in the case of BY and CZ, we have to take these abscissæ, respectively, in the directions BC, CB; CA, AC, when y and z are positive or negative.

For example, if $x = \frac{b}{c}$, then $AX = c = AB$, *i.e.*, X coincides with B, and $x = \frac{b}{c}$ represents the circle which passes through B, C, and touches the side AB at B. See Fig. 2. On the other hand, if

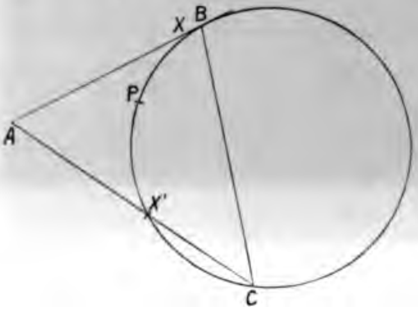

FIG. 2. $x = \frac{b}{c}$. $\angle BXC = \pi - B$, when X coincides with B.

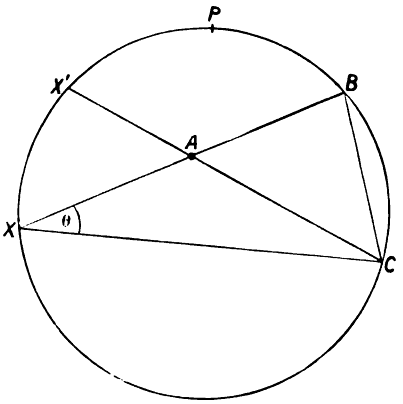

FIG. 3. $x = -\frac{b}{c}$; $AX = -AB = -c$. $x = \frac{\sin\theta}{\sin(\theta - A)} = -\frac{\sin B}{\sin C}$, if $\angle BXC = \theta$.

$x = -\frac{b}{c}$, then $AX = -AB$, so that the circle represented is BXC, where AX is measured off in negative direction BA, and is equal in length to AB or c. See Fig. 3. Again, since $x = \frac{c}{AX'}$, it is clear that $x = \frac{c}{b}$ is the circle (BC) touching AC at C (Fig. 1). We thus easily discover the Brocard points; viz., $\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$ and $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$.

4. The above method of determining the position of a point P, by the cointersection of three circles $x = x_1$, $y = y_1$, $z = z_1$, gives an indeterminate result in two exceptional cases; viz., when (1) x_1, y_1, z_1 are all zero, *i.e.*, if P lies on the line of infinity; (2) all infinite, or P is on the circumcircle ABC. To fix the position of P, in either of these cases, we must know the ratios $x : y : z = \alpha : \beta : \gamma$. The vertices A B, C are special points whose tricircular coordinates may be found as follows. For A we have $x = \infty$ and $bz + cy = a$. Here $x = \infty$ denotes the circumcircle ABC, and $bz + cy = a$ is the point-circle at A. The y and z coordinates are thus indeterminate.

5. More generally, the position of a point P can be determined by the cointersection of three curves. If we denote two rational and integral functions of α, β, γ of the nth and $(n-1)$th degrees, respectively, by U_n and V_{n-1}, and write

$$xU_n = \alpha . V_{n-1}, \qquad (1)$$

$$yU_n = \beta . V_{n-1}, \qquad (2)$$

$$zU_n = \gamma . V_{n-1}, \qquad (3)$$

these equations will represent curves, each of the nth degree, having $n^2 - n$ principal points in common, viz., the intersections of

$$U_n = 0, \quad V_{n-1} = 0.$$

Moreover, the remaining common points of (1), (2), and (3), taken in pairs, will lie on the right lines

$$\frac{\beta}{y} = \frac{\gamma}{z}, \quad \frac{\alpha}{x} = \frac{\beta}{y}, \quad \frac{\gamma}{z} = \frac{\alpha}{x},$$

which meet in the point $\frac{\alpha}{x} = \frac{\beta}{y} = \frac{\gamma}{z}$.

In other words, the curves in question have a radical centre P, and the condition for their cointersection is, evidently, that P lies on any one of them; *i.e.*,

$$U_n(x, y, z) = V_{n-1}(x, y, z).$$

In this case also a linear equation

$$lx + my + nz = k$$

represents a curve of the nth degree passing through the $n^2 - n$ principal points. It is evident that x is a function of an abscissa AX, or intercept on the line AB; and, similarly, y, z of BY, CZ, respectively.

Let us consider the curve (1), for instance. This passes through n^2 given points, viz., the $n^2 - n$ principal points and the n intersections of BC with U_n. One more point, X, will therefore completely determine the curve. We may assume X to be on AB. Hence x is a rational function of AX. In tricircular coordinates, as expounded above, we have seen that $x = \frac{b}{AX}$.

We may sum up, briefly, the results of the present note by saying that here a curve of the nth degree, drawn through $n^2 - n$ principal points, is represented by a linear non-homogeneous equation between three coordinates x, y, z, connected by a fundamental non-homogeneous relation of the nth degree; whereas in the ordinary trilinear method the same curve would be represented by a homogeneous equation of the nth degree in α, β, γ, connected by a linear non-homogeneous relation.

Otherwise—A curve of the nth degree W_n can be represented in various ways by a linear equation

$$lx + my + nz = 1.$$

Suppose W_n to be intersected in $n^2 - n$ points by an arbitrary curve V_{n-1} of the $(n-1)$th order. Through these and the angular points of a triangle ABC a curve U_n can be drawn, whose trilinear equation is

$$U_n(\alpha\beta\gamma) = (l\alpha + m\beta + n\gamma)\, V_{n-1}(\alpha, \beta, \gamma) - W_n(\alpha, \beta, \gamma) = 0.$$

Here l, m, n are known quantities. For instance,

$$l = \frac{W_n(1, 0, 0)}{V_{n-1}(1, 0, 0)}.$$

Hence W_n will be represented by

$$lx + my + nz = 1,$$

if we write $$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{V_{n-1}(\alpha, \beta, \gamma)}{U_n(\alpha, \beta, \gamma)}.$$

These equalities give $U_n(x, y, z) = V_{n-1}(x, y, z)$.

On a Canonical Reduction of Bilinear Forms (Part II.), *with special consideration of Congruent Reductions.*

By T. J. I'A. BROMWICH.

Communicated June 14th, 1900. Received, in revised form, October 3rd, 1900.

The following paper is concerned mainly with reductions in which the coefficients of the substitutions on the two sets of variables are either the same (congruent substitutions) or are conjugate imaginaries. Passing to the details of the paper; I give first a brief *résumé* of the general reduction-process, and add a slight extension of a theorem due to Frobenius. This is then used to reduce a single alternate form by a congruent substitution.

The next and longest division of the paper deals with the simultaneous reduction (by congruent substitutions) of a symmetric and an alternate form. So far as I know, this reduction has only been carried out by Kronecker (in the paper subsequently quoted by the letters *Kr.*); and Kronecker's direct object is the reduction of a single bilinear form by a congruent substitution. Thus Kronecker's reduced forms are not always the simplest from the present point of view. Frobenius (*Berliner Sitzungsberichte*, 1896, p. 7) has shown that, if *any* substitutions P, Q (independent of λ) can be found, such that

$$P(\lambda A-B)\,Q=\lambda C-D,$$

and if, further, A, C be both symmetric or both alternate (the same holding for B, D), then R can be derived from P, Q so that

$$R'(\lambda A-B)\,R=\lambda C-D.$$

Thus, if the invariant-factors of $|\,\lambda A-B\,|$ and $|\,\lambda C-D\,|$ are the same, a *congruent* substitution can be found to transform $(\lambda A-B)$ into $(\lambda C-D)$. But, apparently, this method cannot be extended so as to cover the analogous theory for conjugate imaginary substitutions, which would be applied to a pair of Hermite's forms. I have, accordingly, discussed a direct process for finding R in a way that can be applied in both cases.

The reduced forms so obtained have been applied to prove certain theorems of Herr Alfred Loewy's on automorphic substitutions, both of real quadratic forms and of Hermite's forms.

1. *General Account of the Reduction of two Bilinear Forms.*

Let A, B be the two forms, and let the form reciprocal to $(\lambda A-B)$[*] be

$$(\lambda A-B)^{-1}=\sum_{r=1}^{\alpha} P_{r-1}\lambda^{r-1}+\sum_{r=1}^{\beta} Q_r\lambda^{-r}+\sum_{c}\sum_{r=1}^{\gamma} C_r\,(\lambda-c)^{-r},$$

where $(\lambda-c)$ is a typical factor of the determinant $|\,\lambda A-B\,|$, and α, β, γ are the indices of the first invariant-factors (*Elementartheiler*) of $|\,\lambda A-B\,|$ corresponding to $\lambda=\infty$, $\lambda=0$, $\lambda=c$ respectively. In this expression for $(\lambda A-B)^{-1}$ all the terms P will disappear if $|\,A\,|\neq 0$, and all the Q's if $|\,B\,|\neq 0$.

Multiply up by $(\lambda A-B)$, and we have, in general,

$$AP_r-BP_{r+1}=0=AQ_{r+1}-BQ_r=AC_{r+1}+(cA-B)\,C_r,$$

but

$$E=AQ_1-BP_0+\sum_{c} AC_1.$$

The same equations hold if the order of the products is reversed, thus

$$P_rA-P_{r+1}B=0, \text{ \&c.}$$

From these it follows that

$$\begin{aligned}A&=(AQ_1-BP_0)\,A+\sum_{c} AC_1A\\&=AQ_1A-BP_1B+\sum_{c} AC_1A,\end{aligned}$$

and

$$\begin{aligned}B&=(AQ_1-BP_0)\,B+\sum_{c} AC_1B\\&=AQ_2A-BP_0B+\sum_{c}\,(AC_2A+cAC_1A).\end{aligned}$$

If, now, we consider the form $A(\lambda A-B)^{-1}A$, and expand in powers of $(\lambda-c)$, it is clear that the coefficients of $1/(\lambda-c)$ and $1/(\lambda-c)^2$ will be respectively AC_1A and AC_2A. But Weierstrass[†] and Stickelberger[‡] have shown how to arrange $A(\lambda A-B)^{-1}A$ in a form which can be readily expanded in powers of any factor of

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[*] For the definitions of the product of two forms and of the form reciprocal to a given form, see Frobenius (*Crelle*, Vol. LXXXIV., p. 1); or Muth, *Elementartheiler* (Leipzig, 1899). Short accounts will be found in previous papers by the author (*Proc. Lond. Math. Soc.*, pp. 78, 158, above). E is the unit form $(=\Sigma x_r y_r)$.

[†] *Berliner Monatsberichte*, 1868, p. 310; *Gesammelte Werke*, Vol. II., p. 19.

[‡] *Crelle*, Vol. LXXXVI., 1878, p. 20; this paper will be denoted in future by *St.*, for brevity, as we shall frequently have occasion to quote it.

$|\lambda A-B|$ (see also p. 326 below); and so from each expansion we obtain a group of terms in A and B, viz., AC_1A and $AC_2A+cAC'_1A$.

The same method still holds if $c=0$, for, if $A(\lambda A-B)^{-1}A$ be expanded in powers of λ, the coefficients of $(1/\lambda)$, $(1/\lambda)^2$ are respectively AQ_1A, AQ_2A; which have been proved to form parts of A, B respectively. But the case is rather different for $c=\infty$; to deal with this we may first consider $-B(\lambda A-B)^{-1}B$ and expand in powers of $(1/\lambda)$; then the coefficient of λ and the term independent of λ are $-BP_1A$ and $-BP_0B$ respectively, which form parts of our expressions for A and B.

Another method of dealing with this case depends on expanding $B(\mu B-A)^{-1}B$ in powers of μ and picking out the coefficients of $(1/\mu^2)$, $(1/\mu)$. To see this we note that

$$(\mu B-A)^{-1}=-\frac{1}{\mu}\left(\frac{A}{\mu}-B\right)^{-1}$$

$$=-\frac{1}{\mu}\left[\sum_{r=0}^{a-1}P_r\mu^{-r}+\sum_{r=1}^{\beta}Q_r\mu^r+\sum_c\sum_{r=1}^{\gamma}C_r\mu^r(1-\mu c)^{-r}\right].$$

So, on expanding in powers of μ, we find $-P_1$, $-P_0$ as the coefficients of $(1/\mu)^2$, $(1/\mu)$ in the expansion of $(\mu B-A)^{-1}$; hence the coefficients of $(1/\mu)^2$, $(1/\mu)$ in the expansion of $B(\mu B-A)^{-1}B$ in powers of μ are $-BP_1B$, $-BP_0B$ respectively, which are parts of A and B.

Thus, on the whole, we can find the values of A, B by expanding $A(\lambda A-B)^{-1}A$ in powers of the factors $(\lambda-c)$ of $|\lambda A-B|$ and picking out the coefficients of $1/(\lambda-c)$, $1/(\lambda-c)^2$ for each c; and, in case $|A|=0$, we have also to expand $B(\mu B-A)^{-1}B$ in powers of μ, and pick out the coefficients of $(1/\mu)^2$, $(1/\mu)$.

This modification of Weierstrass's solution of the problem was given in part in a short note (*Proc. Lond. Math. Soc.*, p. 158, above), where more details will be found on the determination of the indices of those invariant-factors of $|\lambda A-B|$ which correspond to the infinite roots if $|A|=0$; it will be sufficient to state here that they are the same as the indices of the invariant-factors of $|\mu B-A|$ to base μ.

In the foregoing it is, of course, assumed that the determinant $|\lambda A-B|$ does not vanish identically, or the reciprocal form $(\lambda A-B)^{-1}$ would not exist. The examination of the so-called "singular" case, when $|\lambda A-B|=0$, will be deferred to § 2.

I proceed to an account of Stickelberger's transformation of

$A(\lambda A-B)^{-1}A$, and I shall give an extension of it, suggested by a theorem for quadratic forms due to Frobenius.

Notation.—I employ the following general scheme of symbols—

$$W_{k-1} = \begin{vmatrix} a_{11}, & \ldots, & a_{1n}, & v'_1, & \ldots, & \overset{k-1}{v}_1, & \eta_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{n1}, & \ldots, & a_{nn}, & v'_n, & \ldots, & \overset{k-1}{v}_n, & \eta_n \\ u'_1, & \ldots, & u'_n, & 0, & \ldots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \overset{k-1}{u}_1, & \ldots, & \overset{k-1}{u}_n, & 0, & \ldots, & 0, & 0 \\ \xi_1, & \ldots, & \xi_n, & 0, & \ldots, & 0, & 0 \end{vmatrix}.$$

If $\eta_1, \ldots, \eta_n$ be replaced by $\overset{k}{v}_1, \ldots, \overset{k}{v}_n$, the value so obtained is called U_k; if $\xi_1, \ldots, \xi_n$ be replaced by $\overset{k}{u}_1, \ldots, \overset{k}{u}_n$, the value is V_k; and, if both sets be replaced in this way, the result is Δ_k (*cf. St.*, § 1).

In these symbols, the u's and v's are arbitrary constants whose indices do not refer to powers, but are to be considered as additional suffixes; the ξ's and η's are linear functions of the variables $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ respectively, whose exact form will appear later.

Then, as proved by Stickelberger (*St.*, § 1),

$$W_k\Delta_{k-1} = W_{k-1}\Delta_k - U_kV_k,$$

for W_{k-1}, Δ_k, U_k, V_k are first minors of W_k corresponding to the four zeros which are complementary to Δ_{k-1}. Thus

$$\frac{W_{k-1}}{\Delta_{k-1}} = \frac{W_k}{\Delta_k} + \frac{U_kV_k}{\Delta_k\Delta_{k-1}},$$

and so

$$\frac{W_0}{\Delta_0} = \sum_{k=1}^{n} \frac{U_kV_k}{\Delta_k\Delta_{k-1}};$$

for clearly $W_n \equiv 0$. This result is a generalization of Darboux's[*] for the case of quadratic forms. Its importance in the present investigation is due to the fact that $(-W_0/\Delta_0)$ is the form reciprocal to $A = \Sigma a_{rs}\xi_r\eta_s$ or is A^{-1}.

Frobenius[†] has shown that two consecutive terms of the series

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[*] *Liouville's Journal*, Vol. XIX. (2me sér.), 1874, p. 347 [p. 354, formula (17)].

[†] *Berliner Sitzungsberichte*, 1894, pp. 241, 407; reprinted in *Crelle*, Vol. CXIV., 1895, p. 187. The theorem in question is (7) on p. 249 (*SB.*), p. 196 (*Cr.*). *Frobenius* also verifies that the sum $(U_kV_k/\Delta_k\Delta_{k-1}) + (U_{k+1}V_{k+1}/\Delta_{k+1}\Delta_k)$ can be expressed in this form.

$(U_k V_k / \Delta_k \Delta_{k-1})$ can be combined if so desired; this is done by finding an expression for the difference $(W_{k-1}/\Delta_{k-1} - W_{k+1}/\Delta_{k+1})$. Frobenius's investigation relates to quadratic forms; and I shall now extend his method to bilinear forms, making a slight generalization by using bordered determinants in the place of minors.

Applying Sylvester's theorem* (on determinants of minors) in the form given by Frobenius,† we have

$$W_{k+1}\Delta_{k-1}^2 = \begin{vmatrix} (k, k), & -(k, k+1), & V_k \\ -(k+1, k), & (k+1, k+1), & -V'_k \\ U_k, & -U'_k, & W_{k-1} \end{vmatrix},$$

where (r, s) denotes the value of W'_{k-1} when $\xi_1, \ldots, \xi_n$ are replaced by $u_1^r, \ldots, u_n^r$, and $\eta_1, \ldots, \eta_n$ by $v_1^s, \ldots, v_n^s$; so that $(k, k) = \Delta_k$; also U'_k, V'_k are analogous to U_k, V_k, but with $u^{k+1}, \ldots, u_n^{k+1}$ in place of $u_1^k, \ldots, u_n^k$, and a similar change in the v's. To see the correctness of this equation we have only to notice that the elements of the determinant so written are all second minors of W_{k+1}, corresponding to the last nine zeros of W_{k+1}; and these zeros are complementary to Δ_{k-1}. We have also

$$\Delta_{k-1}\Delta_{k+1} = \begin{vmatrix} (k, k), & -(k, k+1) \\ -(k+1, k), & (k+1, k+1) \end{vmatrix}$$

for (k, k), $-(k, k+1)$, ... are first minors of Δ_{k+1}.

Thus, expanding the determinants, we find

$$W_{k+1}\Delta_{k-1}^2 = W_{k-1}\Delta_{k-1}\Delta_{k+1} - (k+1, k+1)\, U_k V_k - (k, k)\, U'_k V'_k$$
$$+ (k+1, k)\, U'_k V_k + (k, k+1)\, U_k V'_k;$$

or, dividing by $\Delta_{k-1}^2 \Delta_{k+1}$,

$$\frac{W_{k-1}}{\Delta_{k-1}} - \frac{W_{k+1}}{\Delta_{k+1}} = \frac{1}{\Delta_{k-1}^2\Delta_{k+1}}\Big[(k+1, k+1)\, U_k V_k + (k, k)\, U'_k V'_k$$
$$- (k+1, k)\, U'_k V_k - (k, k+1)\, U_k V'_k\Big],$$

which is the extension of Frobenius's theorem.

We may accordingly replace the two consecutive terms $(U_k V_k/\Delta_k\Delta_{k-1}) + (U_{k+1}V_{k+1}/\Delta_{k+1}\Delta_k)$ in our expression for W_0/Δ_0 by the quantity on the right of the last equation.

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* *Phil. Mag.*, April, 1851.

† In § 1 of the paper just quoted; or *Crelle*, Vol. LXXXVI., 1878, p. 44, § 3.

Special cases of this theorem have been given also by Kronecker* and Gundelfinger† for quadratic forms.

For our future investigations we shall employ these results in general when a_{rs} is replaced by $(\lambda a_{rs} - b_{rs})$; and U_k, V_k, Δ_k, W_k will generally be used to mean their values when this change is made in them also, and $(\xi_1, \ldots, \xi_n)$, $(\eta_1, \ldots, \eta_n)$ are replaced by $\left(\frac{\partial A}{\partial y_1}, \ldots, \frac{\partial A}{\partial y_n}\right)$, $\left(\frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_n}\right)$, respectively. For when these substitutions are made we have

$$A(\lambda A - B)^{-1} A = -W_0/\Delta_0 = -\Sigma(U_k V_k)/(\Delta_k \Delta_{k-1}),$$

and in this form we can expand $A(\lambda A - B)^{-1} A$ as explained. It will also be necessary to use the corresponding symbols with $\left(\frac{\partial B}{\partial y_1}, \ldots, \frac{\partial B}{\partial y_n}\right)$, $\left(\frac{\partial B}{\partial x_1}, \ldots, \frac{\partial B}{\partial x_n}\right)$ in place of $(\xi_1, \ldots, \xi_n)$, $(\eta_1, \ldots, \eta_n)$ in order to evaluate the expansion of $B(\mu B - A)^{-1} B$ in powers of μ; but the two investigations are distinct and no confusion need arise.

It will be seen from Stickelberger's paper (*St.*, §§ 1, 2, 5) that in this method the u's and v's cannot be chosen entirely arbitrarily; but that in general the u's and v's may be symmetrical ($u_r^s = v_r^s$) with one special case of exception, when it may happen that for a special value of k we may not have $u_r^k = v_r^k$ in Δ_k; we then avoid Δ_k by using our extension of Frobenius's theorem. For Stickelberger proves that Δ_{k-1} and Δ_{k+1} are not altered by this exception.

2. *Application of Weierstrass's Methods to the "Singular" Case.*

Kronecker (*Berliner Monatsberichte*, March, 1874, p. 156; *Gesammelte Werke*, Bd. I., p. 381) has remarked that the reduction of a "singular" family of bilinear forms could be effected by applying Weierstrass's method to a family obtained by suitably modifying one of the original forms. Kronecker gives the necessary modification, but does not complete the reduction; and, so far as I know, no account of the whole investigation has yet been published.

We have seen in a previous paper (*Proc. Lond. Math. Soc.*, p. 88, above) that, if $| \lambda A - B |$ and all its minors up to the

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* *Berliner Monatsberichte*, 1874, p. 59, § II.; or *Gesammelte Werke*, Vol. I., p. 349. It is given more completely in § 1 of his paper on p. 397 of the *Monatsberichte* (*Werke*, Vol. I., p. 423); the latter paper will be quoted as *Kr*.

† *Crelle*, Vol. XCI., 1881, p. 221, Lemma 2.

$(k-1)$th be identically zero, some of the kth minors not vanishing, then Δ_k (formed by bordering $|\lambda A - B|$ with u's and v's as above explained) will break up into the product of three determinants. Let the two sets of u's and v's be determined so that for

$$u_s^\epsilon = p_s^\epsilon \quad (\epsilon = 1, 2, \ldots, k;\ s = 1, 2, \ldots, n)$$

we have the one determinant* taking the form

$$\begin{vmatrix} \lambda^{\alpha_1}, & 0, & 0, & \ldots, & 0 \\ 0, & \lambda^{\alpha_2}, & 0, & \ldots, & 0 \\ 0, & 0, & \lambda^{\alpha_3}, & \ldots, & 0 \\ \ldots & \ldots & \ldots & & \ldots \\ 0, & 0, & 0, & \ldots, & \lambda^{\alpha_k} \end{vmatrix},$$

while for $u_s^\epsilon = \pi_s^\epsilon$ the determinant takes a similar form with a diagonal of unities.†

In the same way, for $v_s^\epsilon = q_s^\epsilon$ or κ_s^ϵ, we are to have another determinant in the shape

$$\begin{vmatrix} \lambda^{\beta_1}, & 0, & \ldots, & 0 \\ 0, & \lambda^{\beta_2}, & \ldots, & 0 \\ \ldots & \ldots & & \ldots \\ 0, & 0, & \ldots, & \lambda^{\beta_k} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1, & 0, & \ldots, & 0 \\ 0, & 1, & \ldots, & 0 \\ \ldots & \ldots & & \ldots \\ 0, & 0, & \ldots, & 1 \end{vmatrix}.$$

Kronecker's modification of A consists in adding terms; in fact we take

$$C = A + \sum_{\epsilon=1}^{k} t_\epsilon (p_1^\epsilon y_1 + \ldots + p_n^\epsilon y_n)(q_1^\epsilon x_1 + \ldots + q_n^\epsilon x_n),$$

and apply Weierstrass's process to the family $(\lambda C - B)$. The first step is to form the series of determinants $\Delta_1, \Delta_2, \ldots, \Delta_k$ obtained from $|\lambda C - B|$ by bordering it with π's and κ's. We proceed to examine their invariant-factors. To do this we note that

$$\sum_{s=1}^{n} (\lambda c_{rs} - b_{rs}) f_s' = \sum_{s=1}^{n} \left[(\lambda a_{rs} - b_{rs}) + \sum_{\epsilon=1}^{k} \lambda t_\epsilon p_s^\epsilon q_r^\epsilon \right] f_s'$$
$$= \lambda t_1 q_r' (\lambda^{\alpha_1}),$$

---

* This is the determinant in which all the u's appear, when Δ_k is split up into factors; and the second determinant contains all the v's.

† It may be well to remark explicitly that it has been found convenient to alter the notation used in my previous paper (thus, the present p's are the former κ's, &c.).

where f'_s has the same meaning as in the paper already quoted, so that

$$\sum_{s=1}^{n} (\lambda a_{rs} - b_{rs}) f'_s = 0.$$

Similarly, $$\sum_{r=1}^{n} (\lambda c_{rs} - b_{rs}) g'_r = \lambda t_1 p'_s (\lambda^{\beta_1});$$

and analogous results hold for each set of f's and g's.

Using these facts and bordering with the π's, κ's in order, we see that

$$\Delta_0(\lambda)/\Delta_1(\lambda) = -t_1 \lambda^{\alpha_1+\beta_1+1},$$

$$\Delta_1(\lambda)/\Delta_2(\lambda) = -t_2 \lambda^{\alpha_2+\beta_2+1},$$

&c.,

for it must be remembered that in none of these determinants can squares or higher powers of any t appear.* Now, in order that these powers of λ may be the invariant-factors, it is necessary and sufficient that the α's and β's should be arranged in descending order of magnitude.

The next step is to consider the linear functions of x and y which appear in the process of reduction; the function U is found from $\Delta_1(\lambda)$ by putting the linear functions $\frac{\partial C}{\partial y_1}, \ldots, \frac{\partial C}{\partial y_n}$ in the last row in place of $u'_1, \ldots, u'_n$. It is then easy to see that

$$U_1/\Delta_1(\lambda) = f'_1 \frac{\partial C}{\partial y_1} + \ldots + f'_n \frac{\partial C}{\partial y_n} + \text{terms with } t_1 \lambda^{\alpha_1} \text{ as a factor.}$$

Now, owing to the definition of the f's, we see that the expression

$$f'_1 \frac{\partial A}{\partial y_1} + \ldots + f'_n \frac{\partial A}{\partial y_n}$$

---

* For t_1^2 would be multiplied by a determinant, such as

$$\begin{vmatrix} p_r q_g, & p_s q_g \\ p_r q_h, & p_s q_h \end{vmatrix},$$

which vanishes identically.

is only of degree (a_1-1) in λ, not of degree a_1,* so that we may write

$$U_1/\Delta_1(\lambda) = X_1+X_2\lambda+\ldots+X_a\lambda^{a-1}+\text{terms with } t_1\lambda^a \text{ as a factor}$$
$$(a = a_1).$$

Similarly, we shall have

$$V_1/\Delta_1(\lambda) = Y_1+Y_2\lambda+\ldots+Y_\beta\lambda^{\beta-1}+\text{terms with } t_1\lambda^\beta \text{ as a factor}$$
$$(\beta = \beta_1).$$

Thus we shall have, say

$$U_1/\Delta_1(\lambda) = X_1+X_2\lambda+\ldots+X_a\lambda^{a-1}+t_1\lambda^a(X_{a+1}+X_{a+2}\lambda+\ldots),$$
$$V_1/\Delta_1(\lambda) = Y_1+Y_2\lambda+\ldots+Y_\beta\lambda^{\beta-1}+t_1\lambda^\beta(Y_{\beta+1}+Y_{\beta+2}\lambda+\ldots),$$

and then, according to the usual process given by Weierstrass and Stickelberger, we have to pick out the coefficients of $1/\lambda$, $1/\lambda^2$ in the expansion of $-U_1V_1/\Delta_0\Delta_1$; or the coefficients of $\lambda^{a+\beta}$, $\lambda^{a+\beta-1}$ in the product

$$\frac{1}{t_1}\left[X_1+\ldots+X_a\lambda^{a-1}+t_1\lambda^a(X_{a+1}+X_{a+2}\lambda+\ldots)\right]$$
$$\times\left[Y_1+\ldots+Y_\beta\lambda^{\beta-1}+t_1\lambda^\beta(Y_{\beta+1}+\ldots)\right].$$

Hence we find the two sets of terms

(in C)

$$X_1Y_{a+\beta+1}+X_2Y_{a+\beta}+\ldots+X_aY_{\beta+2}+t_1X_{a+1}Y_{\beta+1}+X_{a+2}Y_\beta+\ldots+X_{a+\beta+1}Y_1,$$

(in B) $\quad X_1Y_{a+\beta}+X_2Y_{a+\beta-1}+\ldots+X_aY_{\beta+1}+X_{a+1}Y_\beta+\ldots+X_{a+\beta}Y_1.$

It is easy to see that we find similar sets of terms for each pair of indices a and β; we can now remove the terms in C which are multiplied by $t_1, t_2, \ldots, t_k$, and so obtain the reduced form of A. The comparison of these terms shows that

$$X_{a+1} = q_1'x_1+\ldots+q_n'x_n, \quad Y_{\beta+1} = p_1'y_1+\ldots+p_n'y_n,$$

with similar results for each group of p's and q's.

Finally, it is to be noted that (as proved by Kronecker) the numbers a, β in addition to the ordinary invariant-factors form a *complete* set of invariants.

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* For $\quad f_1'\left(\lambda\frac{\partial A}{\partial y_1}-\frac{\partial B}{\partial y_1}\right)+\ldots+f_n'\left(\lambda\frac{\partial A}{\partial y_n}-\frac{\partial B}{\partial y_n}\right) = 0,$

so that $\lambda\left[f_1'\frac{\partial A}{\partial y_1}+\ldots+f_n'\frac{\partial A}{\partial y_n}\right]$ is only of degree a_1 in λ.

As a consequence of the results that

$$\Delta_0(\lambda)/\Delta_1(\lambda) = -t_1\lambda^{\alpha_1+\beta-1},$$
$$\&c.,$$

it should be observed that the π's, κ's do *not* give values of $\Delta_1(\lambda)$, $\Delta_2(\lambda)$, ..., which are *regular* for the other invariant-factors of $|\lambda C - B|$. These must be treated independently (as is usually the case in dealing with invariant-factors to different bases), and each of them will give a group of terms of the ordinary type found by Weierstrass and Stickelberger.

This general method of dealing with bilinear forms requires very little change for the problem of quadratic forms or symmetrical bilinears. In this respect it differs from Kronecker's last method,* which requires special modifications in dealing with a family of quadratics. Here we have only to note that $\alpha_r = \beta_r$ $(r = 1, 2, ..., k)$, and that each q is equal to its corresponding p, and each κ to π. Then, clearly, each Y is the same function of the y's that the corresponding X is of the x's: and so, in the symmetrical bilinear family, we have sets of terms

$$X_1Y_{2a+1} + \ldots + X_aY_{a+2} + X_{a+2}Y_a + \ldots + X_{2a+1}Y_1,$$
$$X_1Y_{2a} + \ldots + X_aY_{a+1} + X_{a+1}Y_a + \ldots + X_{2a}Y_1;$$

or, in the quadratic forms,

$$2(X_1X_{2a+1} + X_2X_{2a} + \ldots + X_aX_{a+2}),$$
$$2(X_1X_{2a} + X_2X_{2a-1} + \ldots + X_aX_{a+1}).$$

3. *Reduction of a Single Alternate Form to a Canonical Shape by Congruent Substitutions.*

Here the number n of x's and of y's is even; otherwise Δ_0 would vanish. We make the assumption that $u_r^s = -v_r^s$, so that all Δ's are skew-symmetrical determinants, and hence $\Delta_1 = 0 = \Delta_3 = \Delta_5 = \ldots$, while Δ_0, Δ_2, Δ_4, ... are all perfect squares.†

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* *Berliner Sitzungsberichte*, 1890, p. 1225, for bilinear forms; p. 1375, and 1891 pp. 9, 33, for quadratics; from which it will be seen that Kronecker's discussion of the quadratic case is much longer than that of the bilinear.

† Note, the Δ's, &c., are here as originally defined, *i.e.*, without any λ; *e.g.*,

$$\Delta_0 = |a_{rs}|.$$

Suppose then that k is odd; we can apply the preliminary lemma, and we shall have

$$(k,\ k) = 0 = (k+1,\ k+1),$$

for these are both skew-symmetrical determinants of odd order. Also, changing rows into columns and then changing the sign of every row, we see that

$$(k+1,\ k) = -(k,\ k+1) = E_k \text{ say,}$$

and we have $\Delta_{k-1}\Delta_{k+1} = \begin{vmatrix} 0, & (k+1,\ k) \\ (k,\ k+1), & 0 \end{vmatrix} = E_k^2.$

Hence $$\frac{W_{k-1}}{\Delta_{k-1}} = \frac{W_{k+1}}{\Delta_{k+1}} + \frac{1}{\Delta_{k-1}E_k}(U_k'V_k - U_kV_k').$$

Again, taking U_k, changing rows into columns, and changing the signs of every column but the last, we see that U_k and V_k are congruent functions of the ξ's and η's respectively. If, now, we write

$$\xi_r = \frac{\partial A}{\partial y_r}, \quad \eta_r = -\frac{\partial A}{\partial x_r},$$

we shall have that U_k and V_k are congruent functions of the x's and y's; and the same holds for U_k', V_k'.

Making this substitution for the ξ's and η's, we have that

$$W_0 = \Delta_0 A,$$

and so we have $$A = \frac{W_0}{\Delta_0} = \frac{W_2}{\Delta_2} + \frac{1}{\Delta_0 E_1}(U_1'V_1 - U_1V_1'),$$

$$\frac{W_2}{\Delta_2} = \frac{W_4}{\Delta_4} + \frac{1}{\Delta_2 E_3}(U_3'V_3 - U_3V_3'),$$

&c.

Hence A is the sum of $\frac{1}{2}n$ terms of the type just written, or

$$A = \frac{1}{\Delta_0 E_1}(U_1'V_1 - U_1V_1') + \frac{1}{\Delta_2 E_3}(U_3'V_3 - U_3V_3') + \dots$$

$$\dots + \frac{1}{\Delta_{n-2}E_{n-1}}(U_{n-1}'V_{n-1} - U_{n-1}V_{n-1}'),$$

which is the canonical type for an alternate form.* Just as a quadratic form can be reduced to a sum of squares in an infinity of ways, so this can be reduced in an infinity of ways.

Jordan† has given a special case of this, which is obtained by writing

$$u_\alpha^\alpha = 1, \quad u_r^\alpha = 0 \quad (r = 1, 2, \ldots, \alpha - 1, \alpha + 1, \ldots, n).$$

Jordan takes (virtually) the substitution

$$X = -\frac{\partial A}{\partial y_1}, \quad Y = +\frac{\partial A}{\partial x_1}, \quad X' = -\frac{\partial A}{\partial y_2}, \quad Y' = +\frac{\partial A}{\partial x_2},$$

then (if $a_{12} \neq 0$) it is easy to show that the form

$$A - \frac{1}{a_{12}}(XY' - X'Y)$$

is independent of the four variables x_1, y_1, x_2, y_2; and, starting afresh from this form, we can remove four more variables, and so on until the whole form is reduced. It is not difficult to prove by induction that the coefficients of the substitution are all Pfaffians. Thus I find

$$X_1 = (12)\,x_2 + (13)\,x_3 + \ldots + (1n)\,x_n,$$

$$X_2 = (21)\,x_1 + (23)\,x_3 + \ldots + (2n)\,x_n,$$

$$X_3 = \frac{1}{(12)}\left[(1234)\,x_4 + \ldots + (123n)\,x_n\right],$$

$$X_4 = \frac{1}{(12)}\left[(1243)\,x_3 + \ldots + (124n)\,x_n\right],$$

$$X_5 = \frac{1}{(1234)}\left[(123456)\,x_6 + \ldots + (12345n)\,x_n\right],$$

$$X_6 = \frac{1}{(1234)}\left[(123465)\,x_5 + \ldots + (12346n)\,x_n\right],$$

&c.,

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* If n should be odd, it can be readily proved that

$$W_1 = \Delta_1 A,$$

and then we find $\frac{1}{2}(n-1)$ terms of the type just given.

† *Liouville's Journal*, Vol. XIX. (2me sér.), 1874, p. 35.

and then the reduced form of A will be

$$\frac{1}{(12)}(X_1Y_2-X_2Y_1)+\frac{(12)}{(1234)}(X_3Y_4-X_4Y_3)$$

$$+\frac{(1234)}{(123456)}(X_5Y_6-X_6Y_5)+\ldots,$$

where the numbers in brackets denote Pfaffians in the ordinary way; for example,

$$(12)=a_{12},\quad (1234)=a_{12}a_{34}+a_{23}a_{14}+a_{31}a_{24},\ \&c.$$

Frobenius[*] has given a neat proof of Jordan's transformations; and E. von Weber[†] obtains a form equivalent to Jordan's. Muth[‡] has published a short investigation showing how to modify a method of Kronecker's[§] so as to obtain the reduced form of an alternate bilinear form by means of congruent substitutions; this investigation (like Frobenius's) is, in the first place, concerned only with forms whose coefficients are integers (in any assigned region of rationality).

E. von Weber also applies his results to reduce a family of alternate forms; I had worked out this reduction independently (using the transformation above), but shall omit the algebra here.

4. *Simultaneous Reduction of a Symmetric and an Alternate Form by Congruent Substitutions.*

Let A, B be respectively symmetric and alternate; then we must find first the invariant-factors of the determinant $|\lambda A-B|$. By definition of A, B, we have

$$A'=A,\quad B'=-B,$$

where accents refer to the conjugate bilinear forms. Hence

$$\lambda A-B=\lambda A'+B',$$

so that $$|\lambda A-B|=|\lambda A'+B'|=|\lambda A+B|,$$

for the determinant of a bilinear form is equal to that of its conjugate.

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[*] *Crelle* (1879), Vol. LXXXVI., p. 146; the special consideration of alternate forms is in § 7 (p. 166).

[†] *Münchener Sitzungsberichte* (1898), p. 369.

[‡] *Crelle* (1900), Vol. CXXII., p. 89.

[§] *Crelle* (1891), Vol. CVII., p. 135.

Thus for every factor $(\lambda-c)^l$ of $|\lambda A-B|$ there will be another $(\lambda+c)^l$.

Further, if c be complex (and if the coefficients of A and B are *real*), there will be two corresponding conjugate complex roots c_0, $-c_0$ (where c_0 is the complex quantity conjugate to c); but, if $c_0=-c$, or if c be a pure imaginary, these two $(c_0, -c_0)$ are the same as the other two $(-c, +c)$. We shall prove now that the invariant-factors of $|\lambda A-B|$ occur (in general) in pairs.

Consider the value of Δ_k when the sign of λ is changed; first interchange rows and columns and then change the signs of the first n rows and the last k columns. Remembering that $a_{rs}=a_{sr}$, $b_{rs}=-b_{sr}$, we see that the new determinant only differs from Δ_k by having the u's and v's interchanged; but Stickelberger has proved (*St.*, § 5) that we may in general take $u_r^\alpha=v_r^\alpha$; and, making this assumption, we see that changing the sign of λ in Δ_k will only multiply Δ_k by $(-1)^{n-k}$.

So, writing $$\Delta_k=(\lambda-c)^l f(\lambda) \qquad [f(c)\neq 0],$$

we have $$(-1)^{n-k}\Delta_k=(-\lambda-c)^l f(-\lambda).$$

Thus Δ_k has factors of the type $(\lambda-c)^l$, $(\lambda+c)^l$ in pairs; and the invariant-factors (which are obtained from the quotients Δ_{k-1}/Δ_k) will occur in pairs of the type $(\lambda-c)^e$, $(\lambda+c)^e$.

This theorem is due to Kronecker [*Kr.*, p. 440 (p. 477)]. We shall now combine the terms in the reduced forms of A, B which correspond to the pair of invariant-factors $(\lambda-c)^e$, $(\lambda+c)^e$. We have to split $A(\lambda A-B)^{-1}A$ into partial fractions; thus we consider only the special fraction $-(U_kV_k)/(\Delta_{k-1}\Delta_k)$, and evaluate the fractions in $1/(\lambda-c)$, $1/(\lambda+c)$ which can be obtained from this. As explained before (p. 322), we only require the coefficients of $1/(\lambda-c)$, $1/(\lambda-c)^2$, $1/(\lambda+c)$, $1/(\lambda+c)^2$. To find them we write $\lambda=c+t$, $\lambda=-c+t'$, and expand first in powers of t, then in powers of t'.

We shall take for the present U_k to denote the value of U_k with $\lambda=c+t$; and U'_k its value with $\lambda=-c+t'$. The same notation will be used for Δ_k, V_k, and then we have

$$U'_k=\begin{vmatrix} (-c+t')a_{rs}-b_{rs}, & u_r^\alpha, & u_r^k \\ u_s^\alpha, & 0, & 0 \\ \dfrac{\partial A}{\partial y_s}, & 0, & 0 \end{vmatrix} \quad \begin{pmatrix} r, s=1, 2, \dots, n \\ \alpha=1, 2, \dots, k-1 \end{pmatrix},$$

writing the determinant in the shortened form suggested by Frobenius and Nanson.

Change the signs of the first n rows and then of the last k columns in U'_k; finally, change rows into columns, then (remembering $a_{rs} = a_{sr}$, $b_{rs} = -b_{sr}$)

$$U'_k = (-1)^{n-k} \begin{vmatrix} (c-t')\,a_{rs}-b_{rs}, & u_r^1, & \dfrac{\partial A}{\partial y_r} \\ u_s^1, & 0, & 0 \\ u_s^k, & 0, & 0 \end{vmatrix}.$$

Hence, if we change the x's to y's and write $t' = -t$ in U'_k, we should have

$$U'_k = (-1)^{n-k}\,V_k,$$

and, with a similar change,

$$V'_k = (-1)^{n-k}\,U_k.$$

By the same process we find

$$\Delta'_k = (-1)^{n-k}\Delta_k, \quad \Delta'_{k-1} = (-1)^{n-k+1}\Delta_{k-1},$$

where $t' = -t$.

If, then, we write

$$-\frac{U_k V_k}{\Delta_k \Delta_{k-1}} = \frac{1}{t^e}\,(X_1 + X_2 t + \ldots)(Y_{e+1} + Y_{e+2}t + \ldots),$$

we shall have

$$-\frac{U'_k V'_k}{\Delta'_k \Delta'_{k-1}} = \frac{-1}{(-t')^e}(Y_1 - Y_2 t' + Y_3 t'^2 - \ldots)(X_{e+1} - X_{e+2}t' + \ldots),$$

where Y_r is the same function of the y's that X_r is of the x's. Now the terms we require will come from the coefficients of $1/t$, $1/t^2$, $1/t'$, $1/t'^2$, which will be

$$\begin{gathered} (X_1 Y_{2e} + X_2 Y_{2e-1} + \ldots + X_e Y_{e+1}), \\ (X_1 Y_{2e-1} + X_2 Y_{2e-2} + \ldots + X_{e-1} Y_{e+1}), \\ (X_{2e} Y_1 + X_{2e-1} Y_2 + \ldots + X_{e+1} Y_e), \\ -(X_{2e-1} Y_1 + X_{2e-2} Y_2 + \ldots + X_{e+1} Y_{e-1}), \end{gathered}$$

respectively. These will be parts of the coefficients of $1/(\lambda-c)$, $1/(\lambda-c)^2$, $1/(\lambda+c)$, $1/(\lambda+c)^2$ in the expansions of $A\,(\lambda A-B)^{-1}A$ in powers of $(\lambda-c)$, $(\lambda+c)$.

Finally, combining these terms as explained (p. 323), we find the corresponding parts of A, B in the forms

$$(A)\quad X_1Y_{2e}+X_{2e}Y_1+\ldots+X_eY_{e+1}+X_{e+1}Y_e,$$

$$(B)\quad c\,(X_1Y_{2e}-X_{2e}Y_1)+\ldots+c\,(X_eY_{e+1}-X_{e+1}Y_e)$$
$$+(X_1Y_{2e-1}-X_{2e-1}Y_1)+\ldots+(X_{e-1}Y_{e+1}-X_{e+1}Y_{e-1}),$$

where the substitutions are *congruent*.

It may be observed that the substitutions will all be real; and, further, that by means of a similar process, interchanging the parts played by A, B, we can obtain

$$(A)\quad \frac{1}{c}(X_1Y_{2e}+X_{2e}Y_1)+\ldots+\frac{1}{c}(X_eY_{e+1}+X_{e+1}Y_e)$$
$$+X_1Y_{2e-1}+X_{2e-1}Y_1+\ldots+X_{e-1}Y_{e+1}+X_{e+1}Y_{e-1},$$

$$(B)\quad X_1Y_{2e}-X_{2e}Y_1+\ldots+X_eY_{e+1}-X_{e+1}Y_e,$$

as typical terms in the reduced forms.

In case c is a complex quantity, we shall have an exactly similar pair of terms from using c_0 instead of c; if it is desired to restrict ourselves to *real* transformations, we can combine the corresponding terms after first dividing them into real and imaginary parts. If c be a pure imaginary, X_r and X_{e+r} will be conjugate imaginaries.

If c be zero and e be odd, we may still write $u_r^a = v_r^a$ (*St.*, § 5), and in this case there will be no invariant-factor paired off with this one. But we shall have

$$U_k = \begin{vmatrix} \lambda a_{rs}-b_{rs}, & u_r^a, & u_r^k \\ u_s^a, & 0, & 0 \\ \dfrac{\partial A}{\partial y_s}, & 0, & 0 \end{vmatrix},$$

and, treating this as before, we find

$$U_k = (-1)^{n-k}\begin{vmatrix} -\lambda a_{rs}-b_{rs}, & u_r^a, & \dfrac{\partial A}{\partial y_r} \\ u_s^a, & 0, & 0 \\ u_s^k, & 0, & 0 \end{vmatrix},$$

or
$$U_k = (-1)^{n-k}\,V_k,$$

where in V_k we replace λ by $(-\lambda)$ and the y's by x's.

Thus, if $e = 2p+1$, we have to pick out the coefficients of λ^{2p}, λ^{2p-1} in the product

$$(X_1+X_2\lambda+X_3\lambda^2+\dots)(Y_1-Y_2\lambda+Y_3\lambda^2-\dots),$$

where Y_r is the same function of the y's that X_r is of the x's. So we have typical terms

$$(A)\quad X_1Y_{2p+1}-X_2Y_{2p}+\dots-X_{2p}Y_2+X_{2p+1}Y_1,$$

$$(B)\quad (X_{2p}Y_1-X_1Y_{2p})+\dots\pm(X_{p+1}Y_p-X_pY_{p+1}).$$

Now return to the case of exception previously alluded to (p. 326), which occurs when $c=0$ and the index of an invariant-factor is *even;* then (as shown by Stickelberger) it is not permissible to write $u_r^a = v_r^a$, and, to avoid this difficulty, we use the preliminary lemma given above. In the first place, one of the consecutive invariant-factors is the same (*St.*, § 5). Suppose that the invariant-factors (to base λ) given by Δ_{k-1}/Δ_k and Δ_k/Δ_{k+1} are the same; let their common index be e, where e is even. By our lemma we have a part of $A(\lambda A-B)^{-1}A$,

$$-\frac{1}{\Delta_{k-1}^2\Delta_{k+1}}\Big[(k+1,\,k+1)\,U_kV_k+(k,\,k)\,U_k'V_k'$$
$$-(k+1,\,k)\,U_k'V_k-(k,\,k+1)\,U_kV_k'\Big],$$

where in all the determinants we write $(\lambda a_{rs}-b_{rs})$ instead of a_{rs}. We may assume now that $u_r^a = v_r^a$; this will make $(k,\,k)$ and $(k+1,\,k+1)$ divisible by a higher power of λ than appears in *every* kth minor of $|\,\lambda A-B\,|$ (*St.*, § 5); let that which appears in every kth minor be λ^l, so that λ^{l+e} divides every $(k-1)$th minor and λ^{l-e} every $(k+1)$th. Thus, if

$$\Delta_{k-1} = \lambda^{l+e}(\delta_1+\delta_2\lambda+\delta_3\lambda^2+\dots),$$

we have, by changing the sign of λ as above (p. 335),

$$(-\lambda)^{l+e}\left[\delta_1+\delta_2(-\lambda)+\delta_3(-\lambda)^2+\dots\right]$$
$$=(-1)^{n-k+1}\Delta_{k-1}$$
$$=(-1)^{n-k+1}\lambda^{l+e}\left[\delta_1+\delta_2\lambda+\delta_3\lambda^2+\dots\right].$$

But $\delta_1 \neq 0$, or Δ_{k-1} would be divisible by a higher power of λ than λ^{l+e}, and so $(n-k+l)$ is odd (as e is even). Also, from the last equation, $\delta_2 = 0$, $\delta_4 = 0$, &c.; so we may write

$$\Delta_{k-1} = \lambda^{l+e}(\delta_1+\delta_3\lambda^2+\delta_5\lambda^4+\dots).$$

In the same way we have

$$\Delta_{k+1} = \lambda^{l-s}(\epsilon_1+\epsilon_3\lambda^2+\epsilon_5\lambda^4+\ldots).$$

Again, if we change the sign of λ in $(k, k+1)$, we get $(k+1, k)$ multiplied by $(-1)^{n-k}$, and so we write

$$(k, k+1) = \lambda^l(\gamma_1+\gamma_2\lambda+\gamma_3\lambda^2+\ldots)$$
$$(k+1, k) = (-1)^{n-k+l}\lambda^l(\gamma_1-\gamma_2\lambda+\gamma_3\lambda^2-\ldots)$$
$$= -\lambda^l(\gamma_1-\gamma_2\lambda+\gamma_3\lambda^2-\ldots),$$

and, applying the same process to (k, k), we see that, if

$$(k, k) = \lambda^l(a_1+a_2\lambda+a_3\lambda^2+\ldots),$$

then also $\quad (k, k) = -\lambda^l(a_1-a_2\lambda+a_3\lambda^2-\ldots).$

Hence $a_1 = 0$, $a_3 = 0$, ..., and so

$$(k, k) = \lambda^{l+1}(a_2+a_4\lambda^2+a_6\lambda^4+\ldots).$$

Similarly, $\quad (k+1, k+1) = \lambda^{l+1}(\beta_2+\beta_4\lambda^2+\beta_6\lambda^4+\ldots).$

Using the same arguments, we readily see that, if we write

$$U_k = \lambda^l(\xi_1+\xi_2\lambda+\ldots), \qquad U'_k = \lambda^l(\xi'_1+\xi'_2\lambda+\ldots),$$

then $\quad V_k = -\lambda^l(\eta_1-\eta_2\lambda+\ldots), \quad V'_k = -\lambda^l(\eta'_1-\eta'_2\lambda+\ldots),$

where η_r, η'_r are the same functions of the y's as ξ_r, ξ'_r of the x's.

For brevity write

$$(a) = a_2+a_4\lambda^2+\ldots, \text{ \&c.,}$$
$$(\xi) = \xi_1+\xi_2\lambda+\ldots, \text{ \&c.,}$$
$$(\eta) = \eta_1-\eta_2\lambda+\ldots, \text{ \&c.,}$$
$$(\gamma) = \gamma_1+\gamma_2\lambda+\ldots, \quad (\gamma') = \gamma_1-\gamma_2\lambda+\ldots;$$

then we have our typical term in the expression for $A(\lambda A-B)^{-1}A$,

$$-\frac{1}{\lambda^s}\frac{1}{(\delta)^2(\epsilon)}\left[-\lambda(\beta)(\xi)(\eta)-\lambda(a)(\xi')(\eta')+(\gamma)(\xi)(\eta')-(\gamma')(\xi')(\eta)\right].$$

Now put $\quad (\Xi) = (\xi)\left[\frac{(\gamma)}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}}, \quad (\mathrm{H}) = (\eta)\left[\frac{(\gamma')}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}},$

$$(\Xi') = (\xi')\left[\frac{(\gamma')}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}}, \quad (\mathrm{H}') = (\eta')\left[\frac{(\gamma)}{(\delta)^2(\epsilon)}\right]^{\frac{1}{2}},$$

$$(p) = \frac{(\beta)}{[(\gamma)(\gamma)]^{\frac{1}{2}}}, \qquad (q) = \frac{(a)}{[(\gamma)(\gamma')]^{\frac{1}{2}}};$$

then (H), (H′) are derived from (Ξ), (Ξ') respectively, by changing λ to $-\lambda$, and the x's to y's; further, (p), (q) are even functions of λ. With these abbreviations we see that our typical term becomes

$$\frac{1}{\lambda^e}\left[(\Xi')(\mathrm{H})-(\Xi)(\mathrm{H}')+\lambda\,(p)(\Xi)(\mathrm{H})+\lambda\,(q)(\Xi')(\mathrm{H}')\right].$$

It is now possible to determine (g), (h), (g'), (h') power-series in λ^2 so that the expression in square brackets is equal to

$$\left[(g')(\Xi')+\lambda\,(g)(\Xi)\right]\left[(h)(\mathrm{H})-\lambda\,(h')(\mathrm{H}')\right]$$
$$-\left[(h)(\Xi)+\lambda\,(h')(\Xi')\right]\left[(g')(\mathrm{H}')-\lambda\,(g)(\mathrm{H})\right],^*$$

or to
$$(X)(Y')-(X')(Y),$$

where (Y) and (Y') are obtained from (X) and (X') respectively by changing the x's to y's, and λ to $-\lambda$.

Now, if we write

$$(X)=X_1+X_2\lambda+X_3\lambda^2+\ldots,\qquad (Y)=Y_1-Y_2\lambda+\ldots,$$
$$(X')=X_{e+1}-X_{e+2}\lambda+X_{e+3}\lambda^2-\ldots,\quad (Y')=Y_{e+1}+Y_{e+2}\lambda+\ldots,$$

it is clear that Y_r is the same function of the y's as X_r of the x's, and we find that the coefficients of $1/\lambda$, $1/\lambda^2$ in $\frac{1}{\lambda^e}\left[(X)(Y')-(X')(Y)\right]$ are respectively

$$X_1Y_{2e}+X_2Y_{2e-1}+\ldots+X_{2e-1}Y_2+X_{2e}Y_1,$$

$$(X_1Y_{2e-1}-X_{2e-1}Y_1)+(X_2Y_{2e-2}-X_{2e-2}Y_2)+\ldots+(X_{e-1}Y_{e+1}-X_{e+1}Y_{e-1}),$$

and these will be the reduced parts of A, B corresponding to the *pair* of invariant-factors λ^e, λ^e (e even).

In considering the infinite roots of $|\,\lambda A-B\,|=0$, it is easier to treat them as zero roots of $|\,\mu B-A\,|=0$. We use the same notation as before, but with $(\mu b_{rs}-a_{rs})$ in place of $(\lambda a_{rs}-b_{rs})$, and

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* We have to consider an expression of the type $x'y-xy'+axy+bx'y'$, which is equal to $$[(x'+\beta x)(y-\alpha y')-(x+\alpha x')(y'-\beta y)]/(1+\alpha\beta),$$ provided that $$2\beta/a=-2\alpha/a=(1+\alpha\beta).$$ These equations will determine α, β by solving a single quadratic equation. In the special case required α, β, a, b will be power-series containing only *odd* powers of λ.

$\frac{\partial B}{\partial x_r}$, $\frac{\partial B}{\partial y_r}$ in place of $\frac{\partial A}{\partial x_r}$, $\frac{\partial A}{\partial y_r}$, and we are concerned with the invariant-factors to base μ only.

Such invariant-factors may occur singly with an *even* index (*St.*, § 5); but in pairs if the index be odd.* We shall consider first the case when the index is even, then we can treat one invariant-factor by itself; in this case (*St.*, § 5) we may still put $u_r^* = v_r^*$.

We readily see that changing the sign of μ in Δ_k only interchanges the rows and columns, and so does not affect the value of Δ_k; but the corresponding change in U_k gives $(-V_k)$, with the y's changed into x's, for $\frac{\partial B}{\partial x_r} = -\frac{\partial B}{\partial y_r}$ if the y's are changed into x's (as B is alternate). Thus we have (if $e = 2p$) to pick out the coefficients of μ^{2p-2}, μ^{2p-1} in the product

$$-(X_1 + X_2\mu + X_3\mu^2 + \ldots)(Y_1 - Y_2\mu + Y_3\mu^2 - \ldots),$$

where Y_r is the same function of the y's that X_r is of the x's. Hence we find the typical terms

$$(A) \quad -X_1 Y_{2p-1} + X_2 Y_{2p-2} - \ldots + X_{2p-2} Y_2 - X_{2p-1} Y_1,$$

$$(B) \quad (X_1 Y_{2p} - X_{2p} Y_1) - (X_2 Y_{2p-1} - X_{2p-1} Y_2) + \ldots \pm (X_p Y_{p+1} - X_{p+1} Y_p).$$

We have now to consider the case of a pair of invariant-factors of the type μ^e (e odd). The investigation of the corresponding reduced parts of A, B offers little difficulty after the reduction for the pair λ^e, λ^e (e even); we give the similar results without full explanations.

Changing the sign of μ in Δ_{k-1}, Δ_{k+1} will not affect their values, and so we have

$$\Delta_{k-1} = \mu^{l+e}(\delta_1 + \delta_3\mu^2 + \delta_5\mu^4 + \ldots),$$

$$\Delta_{k+1} = \mu^{l-e}(\epsilon_1 + \epsilon_3\mu^2 + \epsilon_5\mu^4 + \ldots),$$

where e is odd and l is odd [instead of $(n-k+l)$ being odd, as before]. Thus we may write

$$(k,\ k+1) = \mu^l(\gamma_1 + \gamma_2\mu + \gamma_3\mu^2 + \ldots);$$

then

$$(k+1,\ k) = -\mu^l(\gamma_1 - \gamma_2\mu + \gamma_3\mu^2 - \ldots)$$

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* This theorem (and the corresponding one relating to the invariant-factors of $|\lambda A - B|$ *to base* λ) are due in the first place to Kronecker [*Kr.*, p. 441 (p. 477)].

by changing the sign of μ. Also

$$(k,\ k) = \mu^{l+1}(\alpha_2+\alpha_4\mu^2+\ldots),$$

$$(k+1.\ k+1) = \mu^{l+1}(\beta_2+\beta_4\mu^2+\ldots).$$

We shall have also $U_k = \mu^l(\xi_1+\xi_2\mu+\xi_3\mu^2+\ldots),$

$$V_k = -(-\mu)^l(\eta_1-\eta_2\mu+\eta_3\mu^2-\ldots)$$

$$= \mu^l(\eta_1-\eta_2\mu+\eta_3\mu^2-\ldots),$$

where η_r is the same function of the y's as ξ_r of the x's. Similar results hold for U'_k, V'_k.

Just as before, we can reduce the expression for

$$(-W_{k-1}/\Delta_{k-1}+W_{k+1}/\Delta_{k+1})$$

to the form $\dfrac{1}{\mu^e}\left[(X)(Y')-(Y)(X')\right].$

where $(X) = X_1+X_2\mu+X_3\mu^2+\ldots,$ $(Y) = Y_1-Y_2\mu+Y_3\mu^2-\ldots,$

$$(X') = X_{e+1}-X_{e+2}\mu+X_{e+3}\mu^2-\ldots,\quad (Y') = Y_{e+1}+Y_{e+2}\mu+Y_{e+3}\mu^2+\ldots,$$

Y_r being the same function of the y's that X_r is of the x's. Hence we find the typical parts

$$(A)\quad X_1Y_{2e-1}+X_2Y_{2e-2}+\ldots+X_{2e-2}Y_2+X_{2e-1}Y_1,$$

$$(B)\quad (X_1Y_{2e}-X_{2e}Y_1)+\ldots+(X_eY_{e+1}-X_{e+1}Y_e).$$

If $|\lambda A-B|\equiv 0$, we may apply the method explained above (§ 2, p. 326). Here we have $\beta_r = \alpha_r$, for, by interchanging the p's and q's and changing the sign of λ, we do not alter the value of $\Delta_k(\lambda)$ (except possibly in sign); hence the invariant-factors belonging to these terms will be of the type λ^{2a+1}, and so (as before, p. 336) we may take each q equal to the corresponding p. Then, by the same investigation as given for the case of invariant-factors to base λ, we find the typical groups of terms

$$(C)\quad X_1Y_{2a+1}-X_2Y_{2a}+\ldots+(-1)^a tX_{a+1}Y_{a+1}+\ldots-X_{2a}Y_2+X_{2a+1}Y_1,$$

$$B)\quad X_1Y_{2a}-X_2Y_{2a-1}+\ldots+X_{2a-1}Y_2-X_{2a}Y_1.$$

Thus we have in the original forms

$$(A)\quad (X_1Y_{2a+1}+X_{2a+1}Y_1)-(X_2Y_{2a}+X_{2a}Y_2)+\dots$$
$$\dots+(-1)^{a-1}(X_aY_{a+2}+X_{a+2}Y_a),$$
$$(B)\quad (X_1Y_{2a}-X_{2a}Y_1)-(X_2Y_{2a-1}-X_{2a-1}Y_2)+\dots$$
$$\dots+(-1)^{a-1}(X_aY_{a+1}-X_{a+1}Y_a),$$

and there are similar sets of terms for each **a**. These **a**'s are Kronecker's *Minimalgradzahlen*, and may be found as described in my former paper (*Proc. Lond. Math. Soc.*, pp. 87–92 and p. 111 above); the invariants used by Kronecker are the numbers $2a+1$, which are the numbers of variables in the sets of reduced terms.

We have now six types of reduced forms, corresponding to five types of invariant-factors and the singular case, when $|\lambda A-B|\equiv 0$. These will be found to be in agreement with Kronecker's results,[*] except in the arrangement of suffixes; there is also a superficial difference in the first and second classes (following Kronecker's order of arrangement, the sixth and first in the foregoing), due to the fact that Kronecker has reduced his results so as to give the neatest type for the bilinear form $(A+B)$. To indicate the real agreement it will be sufficient to consider two special cases—

(a) Corresponding to the case $|\lambda A-B|\equiv 0$, with a *Minimalgradzahl* 2, Kronecker gives the type

$$A+B=x_1y_2+x_2y_3+x_3y_4+x_4y_5,$$

and so $\qquad A-B=A'+B'=x_2y_1+x_3y_2+x_4y_3+x_5y_4.$

Making the substitutions

$$x_1+2x_3+x_5=2X_1,\qquad x_2+x_4=2X_2,$$
$$x_1\qquad -x_5=2X_3,\qquad -x_2+x_4=2X_4,$$
$$x_1-2x_3+x_5=2X_5,$$

with the congruent substitutions for the y's, we find

$$A=X_1Y_2+X_2Y_1-(X_3Y_4+X_4Y_3),$$
$$B=-X_2Y_3+X_3Y_2+X_4Y_5-X_5Y_4,$$

agreeing with our general result.

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[*] See the paper *Kr.*, p. 440 (*Werke*, Bd. I., p. 475); the list is reproduced on p. 146 of Dr. Muth's *Theorie der Elementartheiler*, Leipzig, 1899.

(*b*) Again, Kronecker gives a form

$$A+B = x_1y_2+ax_2y_1+x_2y_3+ax_3y_2+x_3y_4+ax_4y_3,$$

and so $$A-B = x_2y_1+ax_1y_2+x_3y_2+ax_2y_3+x_4y_3+ax_3y_4.$$

If we put $a = (1+c)/(1-c)$ and then make the substitution

$$x_1 = -(X_3+2cX_4), \quad x_3 = \frac{1-c}{1+c}X_3,$$

$$x_2 = -\frac{1-c}{2c}X_1, \qquad x_4 = (1+c)\,X_2,$$

with the congruent substitutions for the y's, we have

$$A = X_1Y_4+X_4Y_1+X_2Y_3+X_3Y_2,$$

$$B = c\,(X_1Y_4-X_4Y_1)+c\,(X_2Y_3-X_3Y_2)+(X_1Y_3-X_3Y_1),$$

agreeing with what has been found before, corresponding to the invariant-factors $(\lambda-c)^2$, $\lambda+c)^2$ of $|\,\lambda A-B\,|$.

5. *Application of the Reduction to Properties of Automorphic Substitutions.*

Herr Alfred Loewy[*] has proved certain propositions relating to automorphic substitutions of a real symmetric bilinear form (or of a quadratic form) of non-zero determinant. These are deduced (in § 9, p. 424, of the paper quoted in the footnote) from similar properties proved for conjugate imaginary substitutions which are automorphic for Hermite's forms (see below, p. 350). The following investigation is complete in itself.

Suppose that S is a real symmetric bilinear form and P a real substitution automorphic for S; then, in Frobenius's symbolical form,

$$P'SP = S.$$

Frobenius has proved[†] that we can find P_0 similar (*ähnlich*)[‡] to P and S_0 congruent with S, so that $P_0'S_0P_0 = S_0$. Further, P_0, S_0 will be each separable (*zerlegbar*) into two parts such that $P_0 = P_1+P_2$, $S_0 = S_1+S_2$, where the variables in P_1, S_1 are the same and do not appear in P_2, S_2, and *vice versa*; finally, $P_1' S_1 P_1 = S_1$, $P_2' S_2 P_2 = S_2$. The invariant-factors of $| P_1 - \lambda E_1 |$ are all those of $| P_0 - \lambda E |$ (or of $| P - \lambda E |$) which have the base $(\lambda + 1)$; and the other invariant-factors of $| P - \lambda E |$ appear as those of $| P_2 - \lambda E_2 |$. Here the unit-form E (*Einheitsform*) has been separated into E_1, E_2 in the same way as P_0, S_0. The process followed in finding P_1, S_1, P_2, S_2 from P, S shows that these are all real and obtained by real transformations. Then we have

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[*] *Nova Acta Leop.-Carol. Akad.*, 1898, Vol. LXXI., p. 379; an abstract appeared in *Math. Annalen*, Vol. L., p. 557.

[†] *Crelle*, 1878, Vol. LXXXIV., p. 1; this proposition is given in § 10.

[‡] Two forms P, P_0 are called *ähnlich* by Frobenius, if a third form Q can be found such that $P_0 = QPQ^{-1}$.

$$P_1 = -(S_1 + T_1)^{-1}(S_1 - T_1), \quad P_2 = +(S_2 + T_2)^{-1}(S_2 - T_2),$$

where T_1, T_2 are real alternate forms, each containing only the appropriate set of variables.

It follows that $| P_1 - \lambda E_1 |$ has the same invariant-factors as $| \mu_1 S_1 - T_1 |$ $[\mu_1 = (1+\lambda)/(1-\lambda)]$, and $| P_2 - \lambda E_2 |$ the same as $| \mu_2 S_2 - T_2 |$ $[\mu_2 = (1-\lambda)/(1+\lambda)]$.

Now, by hypothesis, $| P_1 - \lambda E_1 |$ has only invariant-factors of the type $(1+\lambda)^e$; so that $| \mu_1 S_1 - T_1 |$ has only those of the type μ_1^e. Thus, if e be even, we have a pair of equal invariant-factors, and the corresponding part of S_1 is (p. 339)

$$x_1 y_{2e} + x_2 y_{2e-1} + \ldots + x_{2e} y_1,$$

and the *signature** of this is zero; while, if e be odd, we have a single invariant-factor giving (see p. 337)

$$x_1 y_e + x_2 y_{e-1} + \ldots + x_e y_1,$$

and the absolute value of the signature is unity. It follows that the signature of S_1 is not greater (in absolute value) than the number of *odd* indices of invariant-factors $(1+\lambda)^e$ of $| P_0 - \lambda E |$.

Now take S_2; we have here five cases—

(i.) The pair of invariant-factors $(\mu_2 - c)^e$, $(\mu_2 + c)^e$ ($c \neq 0$ and *real*), corresponding to $(\lambda - b)^e$, $(\lambda - 1/b)^e$ ($b^2 \neq 1$, and b real).

The corresponding part of S_2 will be

$$x_1 y_{2e} + x_2 y_{2e-1} + \ldots + x_{2e} y_1,$$

which has signature zero.

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* Frobenius defines the *signature* as the number of positive squares, less the number of negative squares, when the form is reduced to squares by *real* transformations.

(ii.) μ_2^e, μ_2^e (e even) corresponding to $(\lambda-1)^e$, $(\lambda-1)^e$. Here the reduced part of S_2 is the same as in (i.).

(iii.) μ_2^e (e odd) corresponding to $(\lambda-1)^e$. Here the reduced part of S_2 is

$$x_1 y_e + x_2 y_{e-1} + \dots + x_e y_1,$$

with the absolute value of the signature equal to unity.

(iv.) $(\mu_2-c)^e$, $(\mu_2+c)^e$, $(\mu_2-c_0)^e$, $(\mu_2+c_0)^e$ (c imaginary, with c_0 for its conjugate, $c+c_0 \neq 0$) corresponding to $(\lambda-b)^e$, $(\lambda-1/b)^e$, $(\lambda-b_0)^e$. $(\lambda-1/b_0)^e$. We find for S_2 the part

$$(x_1 y_{2e} + \dots + x_{2e} y_1) + \text{the conjugate imaginary},$$

and we can readily reduce this to a *real* form, whose signature is seen to be zero.

(v.) $(\mu_2-c)^e$, $(\mu_2+c)^e$ (c pure imaginary), corresponding to $(\lambda-b)^e$. $(\lambda-1/b)^e$ (mod $b = 1$). This gives for S_2

$$x_1 y_{2e} + x_2 y_{2e-1} + \dots + x_{2e} y_1,$$

and, from the method used in making the reduction in the last section. x_{e+r} and x_r are conjugate imaginaries. It is now easy to see that the signature of this part of S_2 is zero if e be even, or 2 if e be odd. It follows that the absolute value of the signature of S_2 is not greater than the number of odd indices of invariant-factors of $| P_2-\lambda E |$ of the type $(\lambda-b)^e$ (mod $b = 1$).

Hence, finally, as the absolute value of the signature of $S_0 = S_1 + S_2$ is not greater than the sum of the absolute values of the signatures of S_1 and S_2, it follows that the absolute value for S_0 is not greater than the number of odd indices of invariant-factors of $| P_0-\lambda E |$ of the type $(\lambda-b)^e$ (mod $b = 1$, including $b = \pm 1$). But, since S_0 is derived from S by a *real* substitution, their signatures are equal; and, since P, P_0 are similar, the invariant-factors of $| P-\lambda E |$, $| P_0-\lambda E |$ are the same. Hence the theorem follows:

If P be a real substitution which is automorphic for the real quadratic form S, the signature of S is not greater in absolute value than the number of odd indices of invariant-factors of $| P-\lambda E |$, only those of the type $(\lambda-b)^e$ (mod $b = 1$) being counted.

This theorem is equivalent to the inequality in § 9 of Herr Loewy's paper. Herr Loewy asks me to state that this form of the theorem was familiar to him when his paper was published; but that, for the applications he had in view (§§ 10–12), the form of the theorem given in his paper was found more suitable; this form of the theorem can be deduced immediately from the above by introducing the *characteristic* (as defined by Loewy)* of S. A similar remark applies to the theorem stated on p. 163 above.

6. *Simultaneous Reduction of two Hermite's Forms to a Canonical Shape.*

A *Hermite's form* is a bilinear form in which the coefficients a_{rs}, a_{sr} are conjugate imaginaries; and the variables x_r, y_r are also conjugate imaginaries, but, for purposes of symbolical calculation, it is often convenient to leave this out of reckoning, just as we frequently use symmetrical bilinear forms instead of quadratic forms.

Symbolically, a Hermite's form (A) is defined by the condition

$$A' = A_0$$

where the suffix 0 indicates that we should take the conjugate imaginary of each *coefficient* (the effect of the suffix is not to be extended to the variables).

If A, B are Hermite's forms to be reduced to canonical shapes, we are restricted to conjugate imaginary substitutions on the x's and the y's; we have thus, in Frobenius's notation, to determine a form S which will give

$$S_0'AS, \quad S_0'BS$$

as reduced forms.

We shall see that we can modify Stickelberger's results (*St.*, § 1) so as to reduce A, B by substitutions of the type considered. We show, first, that the invariant-factors of $|\lambda A - B|$ occur in pairs of the form $(\lambda - c)^e$, $(\lambda - c_0)^e$, where c, c_0 are conjugate imaginaries.

For, according to Stickelberger (*St.*, § 5), we may always choose our u's and v's so that u_r^*, v_r^* are conjugate complex quantities; in fact, they may be real and equal except in two special cases. With this assumption, take the conjugate imaginary of Δ_k (without changing λ) and change rows into columns; we then have Δ_k once more. Hence

$$\Delta_k = (\Delta_k)_0.$$

Thus, if $$\Delta_k = (\lambda - c\ f(\lambda) \qquad [f(c) \neq 0],$$

we have $$\Delta_k = (\lambda - c_0)^e f_0(\lambda) \qquad [f_0(c_0) \neq 0].$$

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* *The characteristic* of a quadratic form is the number of positive or negative *squares in the reduced form*; the *smaller* of these numbers being taken.

It follows that factors of Δ_k of the type $(\lambda-c)^l$, $(\lambda-c_0)^l$ always appear in pairs, and thus, since the invariant-factors are given by the series of quotients Δ_{k-1}/Δ_k, the invariant-factors also appear in pairs of the type $(\lambda-c)^e$, $(\lambda-c_0)^e$. If c is real, or $c=c_0$, this result will no longer hold.

Next we combine the parts of the fraction $-(U_kV_k)/(\Delta_k\Delta_{k-1})$ which are obtained from the invariant-factors $(\lambda-c)^e$, $(\lambda-c_0)^e$. Let us write U_k for its value when $\lambda=c+t$, U'_k for its value when $\lambda=c_0+t'$; then, taking the conjugate imaginary of U'_k (without altering the x's or t'), we find, changing rows into columns,

$$(U'_k)_0 = \left.\begin{array}{ccc} (c+t')\,a_{rs}-b_{rs}, & v_r^a, & \dfrac{\partial A_0}{\partial y_r} \\ u_s^a, & 0, & 0 \\ u_s^k, & 0, & 0 \end{array}\right| \quad \begin{pmatrix} r,\, s = 1,\, 2,\, \ldots,\, n \\ a = 1,\, 2,\, \ldots,\, k-1 \end{pmatrix}.$$

Now $\dfrac{\partial A_0}{\partial y_r}$ is the same function of the x's as $\dfrac{\partial A}{\partial x_r}$ is of the y's, by definition of A. Hence, if t' be changed to t in $(U'_k)_0$, we obtain the same function of the x's as V_k is of the y's. Similar results hold for $(V'_k)_0$ and U_k. Hence we may write

$$-\frac{U_kV_k}{\Delta_k\Delta_{k-1}} = \frac{1}{t^e}(X_1+X_2t+\ldots)(Y_{e+1}+Y_{e+2}t+\ldots)$$

and
$$-\frac{U'_kV'_k}{\Delta'_k\Delta'_{k-1}} = \frac{1}{t'^e}(Y_1+Y_2t'+\ldots)(X_{e+1}+X_{e+2}t'+\ldots),$$

where the X's are derived from the x's and the Y's from the y's by conjugate imaginary substitutions, by virtue of what has been proved connecting U_k and V'_k, U'_k and V_k, Δ_k and Δ'_k.

Combining the coefficients of $1/t$, $1/t^2$, $1/t'$, $1/t'^2$ (as explained previously), we find the reduced parts

$$(A)\quad X_1Y_{2e}+X_2Y_{2e-1}+\ldots+X_{2e}Y_1,$$

$$(B)\quad c\,(X_1Y_{2e}+\ldots+X_eY_{e+1})+c_0\,(X_{e+1}Y_e+\ldots+X_{2e}Y_1)$$
$$+X_1Y_{2e-1}+X_2Y_{2e-2}+\ldots+X_{2e-1}Y_1.$$

If $c=c_0$, we do not have a pair of invariant-factors in general; but, writing $\lambda = c+t = c_0+t$, we find that

$$(U_k)_0 = \begin{vmatrix} (c+t)\,a_{rs}-b_{rs}, & v^a_r, & \dfrac{\partial A_0}{\partial y_r} \\ u^a_s, & 0, & 0 \\ u^k_s, & 0, & 0 \end{vmatrix} \quad \begin{pmatrix} r, s = 1, 2, \ldots, n \\ a = 1, 2, \ldots, k-1 \end{pmatrix}.$$

which is the same function of the x's as V_k is of the y's $\left(\text{by virtue of the relation between } \dfrac{\partial A_0}{\partial y_r} \text{ and } \dfrac{\partial A}{\partial x_r}\right)$.

Hence, if we write

$$U_k = t^l\,(\xi_1+\xi_2 t+\ldots),$$

we have

$$V_k = t^l\,(\eta_1+\eta_2 t+\ldots),$$

where the ξ's are derived from the x's and the η's from the y's by conjugate imaginary substitutions. If we write

$$\Delta_k = t^l\,(\delta_1+\delta_2 t+\delta_3 t^2+\ldots),$$

we shall have $(\delta_1+\delta_2 t+\delta_3 t^2+\ldots)_0 = \delta_1+\delta_2 t+\delta_3 t^2+\ldots,$

i.e., $\delta_1, \delta_2, \delta_3, \ldots$ are all real.

So write $$-\Delta_k\Delta_{k-1} = t^{2l+e}\,(a_1+a_2 t+\ldots)^2,$$

and then $a_1, a_2, \ldots$ will be all real.* Thus

$$-\frac{U_k V_k}{\Delta_k\Delta_{k-1}} = \frac{1}{t^e}\left(\frac{\xi_1+\xi_2 t+\ldots}{a_1+a_2 t+\ldots}\right)\left(\frac{\eta_1+\eta_2 t+\ldots}{a_1+a_2 t+\ldots}\right).$$

If we write now

$$X_1+X_2 t+\ldots = (\xi_1+\xi_2 t+\ldots)/(a_1+a_2 t+\ldots),$$

$$Y_1+Y_2 t+\ldots = (\eta_1+\eta_2 t+\ldots)/(a_1+a_2 t+\ldots),$$

the substitutions for the Y's and X's are still conjugate imaginaries. Accordingly the reduced parts are

(A) $X_1Y_e+X_2Y_{e-1}+\ldots+X_eY_1$,

(B) $c\,(X_1Y_e+X_2Y_{e-1}+\ldots+X_eY_1)+X_1Y_{e-1}+X_2Y_{e-2}+\ldots+X_{e-1}Y_1$.

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* They might be pure imaginaries, but we can avoid this by changing the signs of the resulting terms in A, B; this change is equivalent to removing the factor i from each a.

The case $c=0$ offers no special feature; and the case of infinite roots is treated, as usual, by taking them as zero roots of $|\mu B-A|=0$. We shall thus find reduced parts (for $c=\infty$)

$$(A)\quad X_1Y_{e-1}+X_2Y_{e-2}+\ldots+X_{e-1}Y_1,$$

$$(B)\quad X_1Y_e+X_2Y_{e-1}+\ldots+X_eY_1.$$

It follows that the problem of reducing two Hermite's forms by conjugate imaginary substitutions has been completely solved, except in the singular case when $|\lambda A-B|\equiv 0$.

In the singular case, following out the method given above (§ 2, p. 326), it is easy to see that reduced forms of the types

$$(A)\quad X_1Y_{2a+1}+X_2Y_{2a}+\ldots+X_aY_{a+2}+X_{a+2}Y_a+X_{a+3}Y_{a-1}+\ldots+X_{2a+1}Y_1,$$

$$(B)\quad X_1Y_{2a}+\ldots+X_aY_{a+1}+X_{a+1}Y_a+\ldots+X_{2a}Y_1$$

will appear, where a is a *Minimalgradzahl.*

Collecting all our results, we have the following theorem :—

If A, B are two Hermite's forms, and σ is the absolute value of the signature of B, then

$$\sigma \overset{=}{<} \textit{the number of odd } h\textit{'s} + \textit{the number of even } l\textit{'s},$$

where h is the index of any invariant factor of $|\lambda A-B|$ corresponding to a real root (not zero) and l is the index of an invariant-factor to base λ. If $|A|=0$, we are to include in the h's the indices of invariant-factors corresponding to infinite roots of $|\lambda A-B|$, such indices being determined as those corresponding to zero roots of $|A-\mu B|=0$ (cf. Proc. Lond. Math. Soc., pp. 158–163 above).

This result holds still if $|\lambda A-B|\equiv 0$. In this case, if the k-th minors of $|\lambda A-B|$ are the first which do not all vanish identically, the number of invariant-factors to base λ is $(d'-k)$; and the number corresponding to $\lambda=\infty$ is $(d-k)$, where d, d' are the defects of A, B respectively (i.e., $n-d$, $n-d'$ are the ranks of A, B).

This theorem is slightly extended from one given by Loewy (*Crelle*, Vol. CXXII., 1900, p. 69); the changes are due to the inclusion of the possibilities (i.) $|A|=0$, (ii.) $|\lambda A-B|\equiv 0$. Loewy's proof depends on reducing the Hermite's forms to two real quadratics; but it is interesting to see the connexion with our former reductions.

7. *Simultaneous Reduction of a Hermite's Form and an associated Hermite's Form; with an Application.*

We first define an *associated Hermite's form* (*beigeordnete Hermite'sche Form*) ; B will be such a form if $B' = -B_0$. Bilinear forms of this type have been introduced by Herr A. Loewy in investigating automorphic substitutions for Hermite's forms.*

If A be a Hermite's form, and it is desired to reduce A, B by conjugate imaginary substitutions, we have only to observe that, if $B = iC$, we have $B_0 = -iC_0$, and so $C' = C_0$. Hence C is a Hermite's form, and the problem is solved at once by means of the investigation in the last section. We can, of course, give an independent investigation (which was the method I followed originally), but this is rather longer, and the results do not reduce to so neat a form without extra labour.

The problem of § 4 ought to be deducible as a case of the above; for a real symmetric form is a special case of a Hermite's form, and a real alternate form of an associated Hermite's form. If such a connexion could be pointed out, it would shorten the investigation of § 4 considerably.

The application is to find an alternative proof of the fundamental inequality of § 5 in Loewy's paper. Loewy shows (§ 4) that, if S be an automorphic substitution of a Hermite's form (A), so that

$$S'_0 A S = A,$$

then S can be reduced to the form

$$S = e^{i\phi}(A+B)^{-1}(A-B),$$

where $B'_0 = -B$ (or B is an associated Hermite's form), and ϕ is some real angle.

In order to examine the characteristic equation of S we consider the form $(rE-S)$; now

$$\begin{aligned} rE-S &= (A+B)^{-1}\left[r(A+B)-e^{i\phi}(A-B)\right] \\ &= (A+iC)^{-1}\left[r(A+iC)-e^{i\phi}(A-iC)\right] \qquad (B=iC) \\ &= i\,(r+e^{i\phi})(A+iC)^{-1}(C-sA), \end{aligned}$$

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* *Nova Acta Leop.-Carol. Akad.*, Vol. LXXI., 1898, p. 379, § 4; and *Math. Annalen*, Vol. L., p. 557.

where
$$s = i\,(r-e^{i\phi})/(r+e^{i\phi}).$$

Hence, to investigate the invariant-factors of $| rE-S |$, we have only to find those of $| sA-C |$, where C is another (arbitrary) Hermite's form; corresponding to an invariant-factor $(s-c)^e$ there will be one $(r-b)^e$, where b, c are connected by
$$c = i\,(b-e^{i\phi})/(b+e^{i\phi})$$
or
$$b = e^{i\phi}\,(1-ic)/(1+ic).$$

It follows that, if b' corresponds to c_0, we have
$$b' = e^{i\phi}\,(1-ic_0)/(1+ic_0).$$

Hence, taking the conjugate imaginary,
$$b'_0 = e^{-i\phi}\,(1+ic)/(1-ic)$$
or
$$bb'_0 = 1,$$
i.e.,
$$b' = 1/b \quad\text{or}\quad b' = 1/b_0.$$

But, if c be real, $b' = b$, since $c_0 = c$, and thus $bb_0 = 1$ or mod $b = 1$.

We conclude that invariant-factors of the type $(r-b)^e$, $(r-1/b_0)^e$ occur in pairs, unless mod $b = 1$. Corresponding to the pair of invariant-factors $(s-c)^e$, $(s-c_0)^e$, we found a part of A of the type
$$x_1 y_{2e} + x_1 y_{2e-1} + \dots + x_{2e} y_1.$$

Taking the pair of terms $x_1 y_{2e} + x_{2e} y_1$, we can write them in the form
$$\tfrac{1}{2}\left[(x_1+x_{2e})(y_1+y_{2e}) - (x_1-x_{2e})(y_1-y_{2e})\right],$$
and similarly for every other pair.

It follows that the *characteristic** of this part of A is e; and the *signature** is zero.

Next, if c be real, we have a part of A, corresponding to $(r-b)^e$, when mod $b = 1$,
$$x_1 y_e + x_2 y_{e-1} + \dots + x_e y_1,$$

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* Loewy (§ 5) defines the *characteristic* (q') of a Hermite's form G as the smaller of the two numbers q, $(n-q)$, when G is put in the form
$$G = \sum_{\alpha=1}^{q} x_\alpha y_\alpha - \sum_{\alpha=q+1}^{n} x_\alpha y_\alpha$$
(the variables being conjugate imaginaries). Frobenius defines the *signature* (σ) of G as $(2q-n)$; so that $2q' = n - \text{mod}\,\sigma$.

for which $q' = \frac{1}{2}e$ or $\frac{1}{2}(e-1)$ and mod $\sigma = 0$ or 1, according as e is even or odd. Now the characteristics of a sum of sets of terms is not less than the sum of the characteristics of the sets separately; while the absolute value of the signature is not greater than the sum of the absolute values of the signatures of the separate sets. *Hence, combining our results, we have that*

$$q \geqq s + \Sigma E\left(\tfrac{1}{2}h\right),$$

$$|\sigma| \overset{=}{<} p,$$

where $2s$ *is the sum of the indices of all the invariant-factors of* $|\, rE - S \,|$ *which vanish for values of* r *whose absolute values are not unity;*[*] h *represents the index of any invariant-factor which vanishes for a value of* r *whose absolute value is unity,* $E\left(\frac{1}{2}h\right)$ *is the greatest integer contained in* $\frac{1}{2}h$. *Further,* p *is the number of the odd* h*'s.*

The above is Loewy's fundamental inequality (§ 5); it is, of course, assumed that (A), the determinant of A, does not vanish; or we should have to consider some further possibilities (as on p. 349). This proof was originally sketched out in a letter from me to Herr Loewy (April, 1900); but the proof of the reduced form of A which I gave there was insufficient. Herr Loewy is publishing a short note[†] in which he proves his inequality by combining my suggestion with the results of a recent paper of his own (*Crelle*, Vol. CXXII., 1900, p. 53).

The theorem in § 7 of Loewy's paper becomes almost intuitive by this method of investigation. This theorem is to the effect that, if we are given a set of invariant-factors satisfying the above inequality and a Hermite's form A, then we can find a substitution which (with its conjugate imaginary) is automorphic for A, and whose characteristic equation possesses the assigned invariant-factors.

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[*] These invariant-factors occur in pairs with equal indices; so that the sum of their indices is an even number $2s$.

[†] *Göttinger Nachrichten*, June 30th, 1900.

Notes on the Theory of Automorphic Functions (*continued*). *By* A. C. Dixon.

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The present paper is supplementary to one published in the Society's *Proceedings* (Vol. xxxi., pp. 297–314), the object of which was to derive the fundamental theorems in Riemann's theory of the Abelian integrals from the properties of Poincaré's theta-Fuchsian series. It appeared to me that, if our knowledge of the properties of those series were complete, we should be able to deduce these theorems from them. At one point (pp. 307, 308) I had to fall back on the older methods to cover a want of rigour in the argument, and my purpose now is to supply this defect.

The results to be proved are numerical; there are three numbers involved:—

p, where $p+1$ is the least number of poles arbitrarily assigned that a Fuchsian function can have, if it is to have no more;

q_1, the number of Abelian integrals of the first kind, or of theta-Fuchsian functions of index 1 without poles;

g, the number of irreducible closed circuits on the closed surface formed from the polygon by fastening together corresponding sides.*

It is to be shown that $p = q_1 = \frac{1}{2}g$.

§§ 1–13 are taken up in proving that $p \not> q_1$. For this it is necessary to discuss the formation, by means of Poincaré's series, of theta-Fuchsian and Fuchsian functions with assigned poles. Particular consideration has to be given to the case when poles are assigned at the vertices of the polygon, and account has to be taken of the special conventions as to the orders of poles and zeroes at these points. The proof would be incomplete if these points were excluded from its scope.

The next result (§ 14) is that $q_1 \not< g-p$; this was established in my former paper.

In §§ 15–19 the well known bilinear relation connecting the moduli of periodicity of two Abelian integrals of the first kind is used to show that $2q_1 \ngtr g$. The relation is proved in §§ 15, 16, and in §§ 17, 18 it is reduced to the ordinary canonical form which is convenient for the argument of § 19.

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* In my former paper this number was denoted by c.

The results $p = q_1 = \frac{1}{2}g$ are thus established.

§ 20 deals with Poincaré's theorem that all theta-Fuchsian functions of integral index > 1 can be expressed by means of his series.

The paper is concerned almost altogether with functions of the first, second, and sixth families. The circular boundary of the functions is generally taken to have centre 0, radius 1.

1. Let $\phi_m(z, y)$ denote the function

$$\Sigma \frac{1}{z-sy}\left(\frac{dsy}{dy}\right)^m,$$

where s denotes one of the substitutions of a Fuchsian group of the first, second, or sixth family, and the summation extends to all substitutions of the group (compare *Acta Math.*, Vol. I., p. 242). Suppose that

$$sy = \frac{\alpha y+\beta}{\gamma y+\delta},$$

and that for the different substitutions of the group s, α, β, γ, δ are distinguished by suffixes. The index m is taken to be an integer greater than 1.

Then, giving y a fixed value c, we have for any substitution s_1 of the group

$$\phi_m(s_1 z, c) = \Sigma \frac{1}{s_1 z-sc}\left(\frac{dsc}{dc}\right)^m$$

$$= \Sigma \frac{1}{s_1 z-s_1 sc}\left(\frac{ds_1 sc}{dc}\right)^m$$

(by rearrangement of the terms)

$$= \Sigma \frac{(\gamma_1 z+\delta_1)(\gamma_1 sc+\delta_1)}{z-sc}\left(\frac{ds_1 sc}{dsc}\right)^m\left(\frac{dsc}{dc}\right)^m$$

$$= (\gamma_1 z+\delta_1)\, \Sigma \frac{1}{z-sc}\,(\gamma_1 sc+\delta_1)^{-2m+1}\left(\frac{dsc}{dc}\right)^m.$$

Thus $\phi_m(s_1 z, c)\left(\frac{ds_1 z}{dz}\right)^{1-m} - \phi_m(z, c)$

$$= \Sigma \frac{1}{z-sc}\left\{\left(\frac{\gamma_1 z+\delta_1}{\gamma_1 sc+\delta_1}\right)^{2m-1} - 1\right\}\left(\frac{dsc}{dc}\right)^m$$

$$= C_0 + C_1 z + C_2 z^2 + \ldots + C_{2m-2} z^{2m-2},$$

where the coefficients C are theta-Fuchsian functions of c of index m, and without poles. Let q_m be the number of linearly independent theta-Fuchsian functions of index m, without poles, whether expressible as theta-Fuchsian series or not. Let $\theta_m^{(r)}$ denote one of these functions, r being one of the numbers 1, 2, ..., q_m. Then, if any $q_m + 1$ values $c_1, c_2, \ldots, c_{q_m+1}$ are taken within the fundamental polygon, it is possible to choose coefficients $A_1, A_2, \ldots, A_{q_m+1}$ so that

$$\sum_{i=1}^{q_m+1} A_i \theta_m^{(r)}(c_i) = 0 \quad (r = 1, 2, \ldots, q_m).$$

Now $C_0, C_1, \ldots, C_{2m-2}$ are linear combinations of $\theta_m^{(1)}(c), \theta_m^{(2)}(c), \ldots$.

Thus $$\sum_{i=1}^{q_m+1} A_i \phi_m(s_1 z, c_i) = \left(\frac{ds_1 z}{dz}\right)^{m-1} \sum_{i=1}^{q_m+1} A_i \phi_m(z, c_i).$$

This holds when s_1 is any substitution of the group, and therefore

$$\sum_{i=1}^{q_m+} A_i \phi_m(z, c_i)$$

is a theta-Fuchsian function of z of negative index $1-m$ having $q_m + 1$ arbitrary poles $c_1, c_2, \ldots, c_{q_m+1}$, and no others.*

2. By means of the derivatives of the function $\phi_m(z, c)$ theta-Fuchsian functions of negative index with given multiple poles may be constructed, since $\frac{\partial^a}{\partial c^a}\phi_m(z, c)$ has a pole at c of order $a+1$, and since

$$\frac{\partial^a}{\partial c^a}\phi_m(s_1 z, c)\left(\frac{ds_1 z}{dz}\right)^{1-m} - \frac{\partial^a}{\partial c^a}\phi_m(z, c)$$

$$= \frac{d^a C_0}{dc^a} + z\frac{d^a C_1}{dc^a} + \ldots + z^{2m-2}\frac{d^a C_{2m-2}}{dc^a}.$$

If, for instance, there is to be a double pole at c_1 and a simple one at each of the other points $c_2, c_3, \ldots, c_{q_m}$, we must choose coefficients or residues $A_1, B_1, A_2, \ldots, A_q$ so that

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* Compare *Acta Math.*, Vol. I., pp. 245, 246.

$$B_1 \frac{d\theta^{(r)}(c_1)}{dc_1} + \sum_{i=1}^{q_m} A_i \theta^{(r)}(c_i) = 0 \quad (r = 1, 2, \ldots q_m),$$

and then the function

$$B_1 \frac{\partial}{\partial c_1} \phi_m(z, c_1) + \sum_{i=1}^{q_m} A_i \phi_m(z, c_i)$$

will be a theta-Fuchsian function of z of index $1-m$ having its poles as desired.

For a multiple pole of order α at c, we must use the values when $y = c$ of $\phi_m(z, y)$ and its first $\alpha-1$ derivatives with respect to y, or, in other words, the first α coefficients in the expansion of $\phi_m(z, y)$ in ascending powers of $y-c$.

In this way a theta-Fuchsian function of index $1-m$ can be constructed with poles arbitrarily assigned, if the sum of the orders of multiplicity of these poles is q_m+1. The q_m+1 residues must satisfy q_m homogeneous linear equations, so that in general their ratios will be determinate and the function will be defined save as to a constant factor. If the q_m equations are not all independent, there will be two or more functions of index $1-m$ with the assigned poles.

In particular cases one or more of the residues may come out with the value zero, so that some of the assigned poles may disappear or be of lower orders than were assigned.

Hence, *if poles within the polygon are arbitrarily assigned, the sum of whose orders of multiplicity is* q_m+1, *at least one theta-Fuchsian function of negative index* $1-m$ *can be constructed, having no poles other than those assigned, and having none of the assigned poles to a higher order than the assigned order.*

I shall call this *Poincaré's negative construction.*

4. Again, if z is supposed fixed within the polygon, $\phi_m(z, y)$ is a theta-Fuchsian function of y of positive index m, and it has a simple pole at z and no other within the polygon. The derivatives of $\phi_m(z, y)$ with respect to z or the functions

$$\Sigma \frac{1}{(z-sy)^\alpha} \left(\frac{dsy}{dy}\right)^m,$$

where α is a positive integer, are theta-Fuchsian functions of y of index m, having multiple poles at z.

By means of these, *theta-Fuchsian functions of positive index* m (>1) *can be constructed with arbitrary poles within the polygon of any orders and with arbitrary residues.* When the poles and their orders are fixed, the number of arbitrary constant coefficients in such a function is q_m+ the sum of the orders of the poles, for q_m is the number of such functions without poles.

This may be called *Poincare's positive construction.* In it we use the coefficients in a Taylor expansion of $\phi_m(z, y)$ considered as a function of z.

5. Both constructions need special investigation when one of the assigned poles is at a vertex of the polygon. Let c be this vertex, $2\pi/\lambda$ the sum of the angles of the cycle to which it belongs, c' the inverse of c with respect to the fundamental circle.

Then we must discuss the behaviour of $\phi_m(z, y)$ when y or z approaches c. Now $\phi_m(z, y)$ is a theta-Fuchsian function of y of index m, and thus

$$\phi_m(z, y)(y-c)^m(y-c')^m$$

is a uniform function of $\left(\frac{y-c}{y-c'}\right)^\lambda$ (*Acta Math.*, Vol. I., p. 218).

Let
$$\frac{y-c}{y-c'}=\eta, \quad \frac{z-c}{z-c'}=\zeta;$$

then it follows that

$$\phi_m(z, y)=(1-\eta)^{2m}\eta^{-m}\psi(\zeta, \eta^\lambda),$$

where ψ denotes a uniform function.

Since $\phi_m(z, y)$ contains a term $\frac{1}{z-y}$, ψ must be infinite when $\eta^\lambda=\zeta^\lambda$, and must, in fact, behave like

$$\frac{\lambda}{c-c}\,\frac{(1-\zeta)^{2-2m}\zeta^{\lambda+m-1}}{\zeta^\lambda-\eta^\lambda}$$

in the neighbourhood of this value of η^λ. Thus $\phi_m(z, y)$ behaves like

$$\frac{\lambda}{c-c'}(1-\eta)^{2m}(1-\zeta)^{2-2m}\frac{\eta^{a\lambda-m}\zeta^{\lambda-a\lambda+m-1}}{\zeta^\lambda-\eta^\lambda},$$

where a is any integer. Now $\phi_m(z, y)$ does not become infinite when z or y alone approaches c, and hence, if a is such that neither $a\lambda-m$ nor $\lambda-a\lambda+m-1$ is negative, we have here an expression from which $\phi_m(z, y)$ will differ by a finite quantity when z or y approaches c, or when both approach c. The necessary value of a is $\left\{\frac{m}{\lambda}\right\}+1$, where $\{x\}$ denotes the integer next below x.

Now for the negative construction we take the successive coefficients in a Taylor expansion of $\phi_m(z, y)$ as a function of y. Here there is an expansion of the form

$$\phi_m(z, y) = (1-\eta)^{2m}\eta^{a\lambda-m}\left[Z_0+\eta^{\lambda}Z_1+\eta^{2\lambda}Z_2+\dots\right],$$

and the functions thus available are

$$Z_0,\ Z_1,\ Z_2,\ \dots .$$

It is seen at once that these are equivalent to the series of derivatives of $\phi_m(z, y)$ with respect to y when $y = c$, but that in the series of derivatives $\lambda-1$ out of every consecutive λ are useless for our purpose, being either identically zero or else linear combinations of lower derivatives.

For a small value of ζ the function Z_n behaves like

$$\frac{\lambda}{c-c'}\,\zeta^{\lambda-a\lambda+m-1-(n+1)\lambda}\,(1-\zeta)^{2-2m},$$

and so appears to have a pole of order

$$(a+n)\,\lambda-m+1.$$

But, according to convention, this order must be divided by λ, as the pole is to be shared among all the polygons which meet in c. Thus the order is

$$a+n-\frac{m-1}{\lambda}$$

in any one polygon; allowance is made in this number for all the vertices of the cycle to which c belongs.

Now the only orders that a pole at these vertices can have for a theta-Fuchsian function of index $1-m$ (*Acta Math.*, Vol. I., p. 218) are the numbers

$$a+n-\frac{m-1}{\lambda}\quad (n = 0, 1, 2, \dots),$$

where

$$a = 1+\left\{\frac{m}{\lambda}\right\}.$$

Hence, by means of the functions $Z_0, Z_1, Z_2, \dots$, the negative construction is still possible when one of the assigned poles is at c. The order assigned to this pole must be "admissible," that is, it must be one of the numbers $a+n-\frac{m-1}{\lambda}$. The functions used in constructing a function with a pole of admissible order h at c will be

$$Z_0, Z_1, \ldots, Z_{\{h\}},$$

and others that are finite at c. Thus the number of residues at c is $\{h\}+1$. It will be convenient to call this the "rank" of the pole. The rank is the same as the order when the order is integral; otherwise the rank is the integer next above the order.

6. For the positive construction in this case we must take the coefficients in the expansion of $\phi_m(z, y)$ in ascending powers of ζ. Suppose

$$\phi_m(z, y) = (1-\eta)^{2m}\eta^{a\lambda-m}\left[Y_0+Y_1\zeta+Y_2\zeta^2+\ldots\right].$$

Then it is seen from the above work that the first of the series $Y_0, Y_1, Y_2, \ldots$ which becomes infinite when $\eta=0$ is

$$Y_{\lambda-a\lambda+m-1},$$

and that this behaves like $\dfrac{\lambda}{c'-c}\eta^{-\lambda}$.

The first that is infinite to a higher order than this is

$$Y_{2\lambda-a\lambda+m-1},$$

and this contains a term $\dfrac{\lambda}{c'-c}\eta^{-2\lambda}$, and so on.

The infinite terms that occur are multiples of $\eta^{-\lambda}, \eta^{-2\lambda}, \eta^{-3\lambda}, \ldots,$ and a term in $\eta^{-n\lambda}$ occurs first in $Y_{n\lambda-a\lambda+m-1}$.

Hence the functions available for the positive construction are the coefficients of

$$\zeta^{\lambda-a\lambda+m-1},\ \zeta^{2\lambda-a\lambda+m-1},\ \ldots$$

in the expansion of $\phi_m(z, y)$ in ascending powers of ζ. These have poles at c of the orders

$$\frac{m}{\lambda}-a+1,\ \frac{m}{\lambda}-a+2,\ \ldots,$$

the order being in each case the index of $\eta^{-\lambda}$ in the most important term; these are the only admissible orders for a pole. The ranks are 1, 2, 3, ..., respectively. It will be substantially the same thing to use the series

$$\Sigma\frac{1}{(sy-c)^{n\lambda-a\lambda+m}}\left(\frac{dsy}{dy}\right)^m \quad (n=1, 2, 3, \ldots).$$

7. When a proposed pole is at a vertex c which lies on the circular boundary there are special difficulties, since this point is an essential singularity of the functions.

The polygon may be altered, without affecting the group, so that c shall form a cycle by itself. Suppose this done; then the substitution by which one of the sides that meet in c is changed into the other must be parabolic; for, if it were hyperbolic, the vertex c could be abolished, and the polygon would have as a side part of the fundamental circle; so that the functions would no longer have the circular boundary.* Let t be this parabolic substitution, its actual

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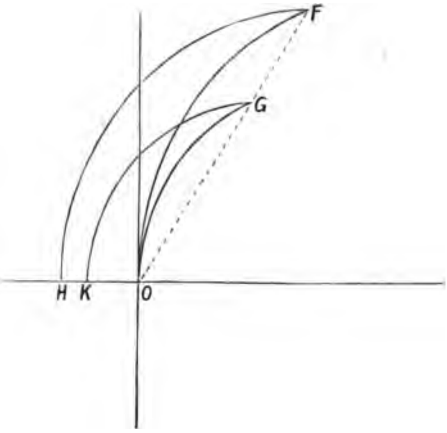
* On this point see Klein's paper (*Math. Ann.*, Vol. XL., pp. 130–139). The conclusion is that an automorphic group may quite well be generated by a polygon with a hyperbolic cycle, but that as a fundamental region this polygon is incomplete. Parts of the area within which the corresponding functions exist are not represented on the polygon, that is, cannot be brought into the polygon by any of the operations of the group.

Take, for instance, a Fuchsian group of real substitutions generated by a polygon, two of whose corresponding sides OF, OG touch at the origin O, F, G being collinear with O, as shown. The substitution that turns OG into OF is of the form



$z' = kz$, where k is real and > 1. Through F, G describe two circular arcs cutting the real axis orthogonally in two points H, K to the left of O, such that $OH = k.OK$. Then the group is unaltered if we add to the polygon the half-meniscus OFH and take away OGK.

Now it is clear that any point between OF and the imaginary axis is represented in the old fundamental polygon by a point in the curvilinear triangle OFG, and that therefore no point on the left of the imaginary axis is represented in the old polygon at all. Poincaré originally concluded from this that the imaginary axis was a natural boundary to the functions generated; but this is not the case, for in the new polygon any point in the second quadrant is represented by a point in the region bounded by HK, the arcs HF, KG, and the imaginary axis. Hence the functions exist in the whole space above the real axis, and, moreover, can be continued across HK, so that they exist in all the plane. In the original polygon the part representing the second quadrant had shrunk up into the point O.

equation being

$$\frac{1}{tz-c} = \frac{1}{z-c} + \frac{2\iota\pi\mu}{c},$$

where μ is real and positive.

Suppose the z-plane transformed by the substitution

$$\zeta = \frac{1}{2\mu}\,\frac{c+z}{c-z},$$

so that the inside of the fundamental circle becomes the half of the ζ-plane on the right of the imaginary axis. Let η, σ, τ correspond to y, s, t, so that

$$\eta = \frac{1}{2\mu}\,\frac{c+y}{c-y},\quad \sigma\zeta = \frac{1}{2\mu}\,\frac{c+sz}{c-sz},\quad \tau\zeta = \frac{1}{2\mu}\,\frac{c+tz}{c-tz} = \zeta - 2\iota\pi,$$

$$\frac{dsy}{dy} = \frac{4c^2}{(y-c)^2}\,\frac{1}{(1+2\mu\sigma\eta)^2}\,\frac{d\sigma\eta}{d\eta} = \left(\frac{1+2\mu\eta}{1+2\mu\sigma\eta}\right)^2 \frac{d\sigma\eta}{d\eta}.$$

Now the substitution τ gives a division of the ζ-half-plane into strips of breadth 2π parallel to the real axis, and each of these strips is further divided by the other operations of the group. One polygon in each strip reaches to infinity, and the sum of the areas of the rest must, therefore, be finite; so that the series

$$\Sigma' \left| \frac{d\sigma\eta}{d\eta} \right|^m$$

must be convergent if $m \not< 2$ and Σ' denotes a summation over those operations which turn the fundamental polygon into another belonging to the same strip. This follows by Poincaré's method, since the areal magnification is $\left| \dfrac{d\sigma\eta}{d\eta} \right|^2$.

Also $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent series. Thus the double series

$$\sum_{n=1}^{\infty} \Sigma' \frac{1}{n^2} \left| \frac{d\sigma\eta}{d\eta} \right|^m$$

is absolutely convergent, and its sum is the product of the sums of the former two.

Now let $f(\eta)$ denote a function of η, uniform, finite, and continuous over the right half-plane and its boundary, and such that

$$\operatorname*{Lt}_{\eta=\infty} \eta^2 f(\eta)$$

is finite or zero. Then, since all the points $\sigma\eta$ lie in the right half-plane, there is a superior limit to the quantities

$$n^2\,|f(\sigma\eta+2n\iota\pi)|,\quad n^2\,|f(\sigma\eta-\overline{2n-2}\,\iota\pi)|\,;$$

and therefore the series

$$\sum_{n=-\infty}^{n=\infty}\Sigma' f(\sigma\eta+2n\iota\pi)\left(\frac{d\sigma\eta}{d\eta}\right)^m,$$

that is,

$$\Sigma f(\sigma\eta)\left(\frac{d\sigma\eta}{d\eta}\right)^m,$$

is absolutely convergent. The convergency is clearly uniform in the domain of an ordinary value of η.

This series represents then a continuous uniform function of η, and, in fact, of $\exp\eta$, since its value is unaffected by the addition of $2\iota\pi$ to η. As η increases $\frac{d\sigma\eta}{d\eta}$ diminishes without limit, except for the identical substitution, and those of the form τ^n. Thus when η is infinite the series reduces to $\operatorname{Lt}_{\eta=\infty}\sum_{n=-\infty}^{n=\infty} f(\eta+2n\iota\pi)$, which is zero if the real part of η is made infinite. Hence when $\exp\eta$ is infinite the function

$$\Sigma f(\sigma\eta)\left(\frac{d\sigma\eta}{d\eta}\right)^m$$

vanishes, and is, in fact, of the same order as $\exp(-\eta)$, or some integral power of this.

Going back to the variable y, we find that

$$\Sigma F(sy)\left(\frac{dsy}{dy}\right)^m \quad\text{or}\quad \left(\frac{2c}{y-c}\right)^{2m}\Sigma\,(2\mu\sigma\eta+1)^{-2m}F\left(c\,\frac{2\mu\sigma\eta-1}{2\mu\sigma\eta+1}\right)\left(\frac{d\sigma\eta}{d\eta}\right)^m$$

represents a theta-Fuchsian function of y, even if the function $F(y)$ has a pole at c, so long as the order of the pole is not higher than $2m-2$. But this theta-Fuchsian function is zero at c to the same order as

$$(y-c)^{-2m}\exp\frac{n}{2\mu}\,\frac{y+c}{y-c}\,.$$

where n is some positive integer; n is, in fact, the order of the zero according to the convention (*Acta Math.*, Vol. I., pp. 216, 217). Thus the positive construction in its ordinary form does not succeed. But it is now readily seen that

$$\Sigma \exp\left(\frac{n}{2\mu}\,\frac{c+sy}{c-sy}\right)\left(\frac{dsy}{dy}\right)^m$$

represents a function suitable for the purpose. For this series may be arranged in the form

$$(1+2\mu\eta)^{2m}\sum_{r=-\infty}^{r=\infty}\Sigma'\left\{1+2\mu\,(\sigma\eta+2r\iota\pi)\right\}^{-2m}\exp n\sigma\eta\left(\frac{d\sigma\eta}{d\eta}\right)^m,$$

and we may suppose the fundamental polygon in the η-plane to be one which reaches to infinity. Then there is a finite superior limit to $\exp n\sigma\eta$ (except when σ is the identical substitution) if η lies in the fundamental polygon, and hence as η increases indefinitely all the terms tend to zero in the aggregate except those of the series

$$(1+2\mu\eta)^{2m}\exp n\eta\sum_{r=-\infty}^{r=\infty}\left\{1+2\mu\,(\eta+2r\iota\pi)\right\}^{-2m}.$$

The sum of these terms is

$$-(1+2\mu\eta)^{2m}\exp n\eta\,\frac{(2\mu)^{-2m}}{(2m-1)!}\left(\frac{d}{d\eta}\right)^{2m-1}\frac{1}{\exp\left(\eta+\frac{1}{2\mu}\right)-1},$$

which is infinite with η to the same order as

$$\eta^{2m}\exp(n-1)\,\eta,$$

or

$$(y-c)^{-2m}\exp\frac{n-1}{2\mu}\,\frac{c+y}{c-y}.$$

Hence the function

$$\Sigma \exp\left(\frac{n}{2\mu}\,\frac{c+sy}{c-sy}\right)\left(\frac{dsy}{dy}\right)^m$$

is available for the positive construction. It has a pole at c whose order is $n-1$, for, according to convention, the order of a pole at c for a theta-Fuchsian function $\theta_m(y)$ of index m is the exponent of $\exp\eta$ in the most important part of $\theta_m(y)(y-c)^{2m}$.

If a pole of order $n-1$ at c is among those assigned to a theta-Fuchsian function of index m which is to be constructed, the functions available, besides those that are finite at c, are n in number, namely,

$$\Sigma \exp\left(\frac{r}{2\mu}\,\frac{c+sy}{c-sy}\right)\left(\frac{dsy}{dy}\right)^m \quad (r = 1, 2, \ldots.\, n).$$

Hence the rank of such a pole is n. This agrees with the general rule that a pole of order h is of rank $\{h\}+1$ if we remember that here properly the order is infinitesimally greater than $n-1$, the integer λ of §§ 5, 6 being infinite.

8. The negative construction is somewhat simpler. The expression $\phi_m(z, y)(y-c)^{2m}$ is a uniform function of $\exp\eta$, and has a simple pole when $y=z$, that is, when $\exp\eta = \exp\zeta$. Thus in the neighbourhood of this pole it behaves like

$$\frac{A}{\exp\eta-\exp\zeta},$$

where
$$A = \operatorname*{Lt}_{y=z} \frac{(y-c)^{2m}}{z-y}(\exp\eta-\exp\zeta)$$
$$= -\frac{c}{\mu}(z-c)^{2m-2}\exp\zeta.$$

Let $\phi_m(z, y)(y-c)^{2m}$ be expanded in descending powers of $\exp\eta$. The successive coefficients will be functions available for the negative construction. The coefficient of $\exp(-n\eta)$ will contain a term

$$-\frac{c}{\mu}(z-c)^{2m-2}\exp n\zeta;$$

so that this coefficient has at c a pole of order n, or, rather, infinitesimally below n.

It is now easily seen that the rank of a pole at c is equal to the order in the case of a theta-Fuchsian function of negative index. For the functions infinite at c that may be used in the negative construction when a pole of order n at c is among those assigned are the coefficients of $\exp(-\eta)$, $\exp(-2\eta)$, ..., $\exp(-n\eta)$, and are therefore n in number.

9. The results reached may now be stated as follows:—By the negative construction *it is possible to form a theta-Fuchsian function of given negative integral index* $-m$, *all of whose poles shall be included among certain points arbitrarily assigned with ranks respectively not exceeding certain positive integers arbitrarily assigned, whose sum is not less than* $q_{m+1}+1$. If this sum is $q_{m+1}+r$, then r such functions can be formed. By the positive construction *it is possible to form a theta-Fuchsian function of given positive integral index* >1 *with any given poles, of any admissible orders, with any assigned residues.*

The rank of a pole of order h is $\{h\}+1$, and it must be borne in mind that the order of a pole at a vertex lying on the circular boundary is infinitesimally above or below an integer according as the index of the function is positive or negative, since the integer λ is infinite for such a vertex. This convention will enable us to use the same numerical formulæ for elliptic and for parabolic vertices, although the proofs of these formulæ may not be the same.

10. With regard to theta-Fuchsian functions of index 1, it is to be noted that none of them can have one simple pole only. This will now be proved.

Let $\theta_1(z)$ be a theta-Fuchsian function of index 1, c one of its poles; then the corresponding residue is

$$\lim_{z=c} (z-c)\,\theta_1(z).$$

Any point sc, into which c is transformed by an operation of the group, is also a pole, and the residue corresponding to sc is

$$\lim_{z=sc} (z-sc)\,\theta_1(z),$$

or

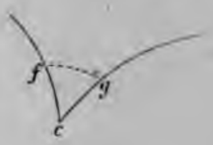
$$\lim_{z=c} (sz-sc)\,\theta_1(sz).$$

This is the same as the residue at c, since

$$\theta_1(sz) = \theta_1(z)\left(\frac{dsz}{dz}\right)^{-1}.$$

Also $\int \theta_1(z)dz$ taken along two corresponding sides of the generating polygon gives equal results and, therefore, when it is taken round the whole perimeter of the polygon, the result is zero; therefore, the sum of the residues of $\theta_1(z)$ at all its poles within the polygon is zero.

11. In the interpretation of this result, the residue at a multiple pole c is, as usual, to be taken as the coefficient of $\dfrac{1}{z-c}$ in the expansion in ascending powers of $z-c$; there are also special conventions relating to the vertices of the polygon when the sum of the angles in a cycle is not 2π. Take an elliptic cycle in which the sum of the angles is $2\pi/\lambda$, and suppose the polygon transformed, if necessary, so that the cycle shall consist of a single vertex c. The expansion of $\theta_1(z)(z-c)(z-c')$, where c' is the inverse of c with respect to the bounding circle, in powers of $\dfrac{z-c}{z-c'}$, contains only such powers as the $h\lambda$-th, where h is any whole number.



Let f, g be two corresponding points near to c on the two sides that meet at c. Suppose f, to be joined by an arc; then it will be necessary to take as the residue of a pole at c the value of

$$-\frac{1}{2\iota\pi}\int_f^g \theta_1(z)\,dz$$

taken along this arc, which must be so near c that the triangle fgc does not include any other pole. Since the angle fcg of the polygon is $2\pi/\lambda$, this residue is $C \div \lambda\,(c-c')$, where C is the absolute term in the expansion of $\theta_1(z)(z-c)(z-c')$ in ascending powers of $(z-c)/(z-c')$.

For, if $h \neq 0$,

$$\int\left(\frac{z-c}{z-c'}\right)^{h\lambda-1}\frac{dz}{(z-c')^2}=\frac{1}{h\lambda\,(c-c')}\left(\frac{z-c}{z-c'}\right)^{h\lambda},$$

which has the same value at f, g; when $h = 0$, we have

$$\int\left(\frac{z-c}{z-c'}\right)^{-1}\frac{dz}{(z-c')^2}=\frac{1}{c-c'}\log\frac{z-c}{z-c'},$$

which is greater at f than at g by $\dfrac{1}{c-c'}\,\dfrac{2\iota\pi}{\lambda}$.

If the vertex c is on the bounding circle and the substitution parabolic, let the substitution that turns cf into cg be

$$\frac{1}{z'-c}=\frac{1}{z-c}-\frac{2\iota\mu\pi}{c},$$

where μ is a real positive quantity. Then $\theta_1(z)(z-c)^2$ in the neighbourhood of c may be expanded in ascending integral powers of $\exp\dfrac{1}{2\mu}\,\dfrac{z+c}{z-c}$. Let the absolute term of this expansion be C. Then

$$-\frac{1}{2\iota\pi}\int_f^g \theta_1(z)\,dz=-\frac{\mu}{c}C,$$

since any integral power of $\exp\dfrac{1}{2\mu}\,\dfrac{z+c}{z-c}$ has the same value at f and g. Thus $-\mu C/c$ must be taken as the residue in this case.

With these special conventions the sum of the residues is zero, and therefore, if there is only a simple pole, its residue must vanish, or a theta-Fuchsian function of index 1 with only a simple pole cannot exist.* A pole of rank 1 is to be counted as simple.

12. It follows from the positive construction that the number of arbitrary coefficients in a theta-Fuchsian function of given index $m\,(>1)$ having given poles of assigned ranks is

$$q_m + \text{the sum of the ranks.}$$

We may hence find an expression for $q_m - q_{m-1}$ when $m > 2$. Take a particular theta-Fuchsian function of index 1, η_1, and let θ_m denote the most general theta-Fuchsian function of index m without poles, ϑ_{m-1} the quotient θ_m/η_1, so that ϑ_{m-1} is a theta-Fuchsian function of index $m-1$. The number of arbitrary coefficients in ϑ_{m-1} is the same as in θ_m, that is, q_m. Now the zeroes and poles of η_1 are poles and zeroes of ϑ_{m-1} in general, but the vertices of the polygon again need special consideration. Suppose the vertex c to be a zero of order $h - \frac{1}{\lambda}$ of η_1, h being a positive integer. Then, since θ_m has a zero at c of order at least

$$\left\{\frac{m}{\lambda}\right\} + 1 - \frac{m}{\lambda},$$

ϑ_{m-1} has a pole of order

$$h + \frac{m-1}{\lambda} - \left\{\frac{m}{\lambda}\right\} - 1$$

at most; the rank of this pole is

$$h + \left\{\frac{m-1}{\lambda}\right\} - \left\{\frac{m}{\lambda}\right\},$$

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* We have here a reason for not expecting theta-Fuchsian series of index 1 in the first, second, or sixth family to converge absolutely. For, if they did, the function $\phi_1(c, z)$ of § 1 would exist, and would be a theta-Fuchsian function of z of index 1 with a simple pole at c and no other, in contravention of the theorem here proved. The same argument applies also to theta-Kleinian series when any of the regions within which the corresponding functions exist is only of finite extent. Another proof is given by Ritter (*Math. Ann.*, Vol. XLI., p. 58). Lindemann, on the other hand, has tried to prove the contrary (*Münchener Sitzungsberichte*, Vol. XXIX., pp. 423–454).

It is possible that such series may converge conditionally; for the two series

$$\phi_1(c, s_1 z)\frac{ds_1 z}{dz} \quad\text{and}\quad \phi_1(c, z)$$

consist of the same terms, but in different orders, and, if the convergency is conditional, we cannot conclude that their sums are equal.

which exceeds the order of the zero of η_1 by

$$\left\{\frac{m-1}{\lambda}\right\}-\left\{\frac{m}{\lambda}\right\}+\frac{1}{\lambda}.$$

This formula applies to an ordinary point by taking $\lambda = 1$, and to a parabolic vertex by taking λ infinite, the result in either case being zero.

If, on the other hand, c is a pole of order $h+\frac{1}{\lambda}$, rank $h+1$, for η_1, then for ϑ_{m-1} it is a zero of order

$$h-\frac{m-1}{\lambda}+\left\{\frac{m}{\lambda}\right\}+1.$$

Now, for any theta-Fuchsian function of index $m-1$, it is, if not a pole, a zero of order at least

$$\left\{\frac{m-1}{\lambda}\right\}+1-\frac{m-1}{\lambda}.$$

The order of the zero being here greater than this by

$$h+\left\{\frac{m}{\lambda}\right\}-\left\{\frac{m-1}{\lambda}\right\},$$

the coefficients in ϑ_{m-1} are restricted by this number of conditions, which falls short of the order of the pole by

$$\left\{\frac{m-1}{\lambda}\right\}-\left\{\frac{m}{\lambda}\right\}+\frac{1}{\lambda}.$$

This is the same expression as before, and again it applies also to an ordinary point and to a parabolic vertex.

The number of zeroes of η_1 exceeds the number of its poles by

$$n-1-\Sigma\frac{1}{\lambda},$$

where $2n$ is the number of sides. The zeroes, generally speaking, are poles of ϑ_{m-1}, and increase the number of its arbitrary coefficients; the poles, on the other hand, are generally zeroes of ϑ_{m-1}, and decrease this number. It follows from the discussion just given that the net effect is to raise the number of arbitrary coefficients from q_{m-1}, which it would be if ϑ_{m-1} had no poles and no assigned zeroes, to

$$q_{m-1}+n-1-\Sigma\frac{1}{\lambda}+\Sigma\left[\left\{\frac{m-1}{\lambda}\right\}-\left\{\frac{m}{\lambda}\right\}+\frac{1}{\lambda}\right]$$

or

$$q_{m-1}+n-1+\Sigma\left\{\frac{m-1}{\lambda}\right\}-\Sigma\left\{\frac{m}{\lambda}\right\}.$$

The summations are taken over the different cycles of vertices. This number would have to be increased if any of the restrictions arising from the zeroes that must be assigned to ϑ_{m-1} were necessarily satisfied; but this cannot be, since it would imply that the function $\vartheta_{m-1}\eta_1$ or θ_m could not possibly have the corresponding poles with arbitrary residues, which we know from the positive construction to be untrue.

It follows then that when $m > 2$

$$q_m - q_{m-1} = n-1-\Sigma\left\{\frac{m}{\lambda}\right\}+\Sigma\left\{\frac{m-1}{\lambda}\right\}.$$

This ceases to hold when $m = 2$, since the positive construction is not available for the index 1. The result is, in fact, untrue. The number of arbitrary coefficients in a theta-Fuchsian function of index 1 with poles and ranks assigned is at most

$$q_1 - 1 + \text{the sum of the ranks},$$

since there cannot be just one pole of rank 1. Thus, by the above method, we find that

$$q_2 - q_1 \not> n-2-\Sigma\left\{\frac{2}{\lambda}\right\}+\Sigma\left\{\frac{1}{\lambda}\right\},$$

or, say, $$q_2 - q_1 = n-2-\Sigma\left\{\frac{2}{\lambda}\right\}-\nu,$$

where ν is zero or a positive integer, for $\left\{\frac{1}{\lambda}\right\} = 0$ always.

It follows by summation that

$$q_m = q_1 - 1 + (m-1)(n-1) - \Sigma\left\{\frac{m}{\lambda}\right\} - \nu.$$

13. We can now discuss the formation of Fuchsian functions with assigned poles. Let ϑ_2 be a theta-Fuchsian series of index 2, having a pole of order $\frac{2}{\lambda} - \left\{\frac{3}{\lambda}\right\}$ for each cycle, and one other pole arbitrarily chosen. Thus it cannot be identically zero. It will have

$$2(n-1) - \Sigma\left\{\frac{3}{\lambda}\right\} + 1 \text{ zeroes.}$$

By the negative construction two theta-Fuchsian functions Θ_{-2}, Θ'_{-}·

can be formed of index -2 with q_3+2, that is,

$$q_1+2n-1-\Sigma\left\{\frac{3}{\lambda}\right\}-\nu \text{ arbitrary poles.}$$

The zeroes of Θ_{-2} or Θ'_{-2} or $A\Theta_{-2}+B\Theta'_{-2}$ will be

$$q_1+1-\Sigma\left\{\frac{3}{\lambda}\right\}+\Sigma\frac{2}{\lambda}-\nu$$

in number, each cycle contributing $\dfrac{2}{\lambda}-\left\{\dfrac{3}{\lambda}\right\}$; since $A\Theta_{-2}+B\Theta'_{-2}$ has an arbitrary zero as well as these at the vertices, it follows that $q_1-\nu$ is not negative, so that the poles of Θ_{-2}, Θ'_{-2} are at least as numerous as the zeroes of ϑ_2.

Suppose then that the zeroes of ϑ_2 are all included among the poles assigned to Θ_{-2}, Θ'_{-2}, and take Θ'_{-2} to be the reciprocal of ϑ_2, which is allowable. Then the product $\Theta_{-2}\vartheta_2$ will not be constant, but will be a Fuchsian function having $q_1-\nu+1$ arbitrary poles, namely, the arbitrary pole of ϑ_2 and the $q_1-\nu$ poles that are still to be assigned for Θ_{-2}, and no others.

In this result, which depends on a combination of the positive and negative constructions, the poles assigned may be at the vertices as well as anywhere else, the proof needing very little modification for this case. The rank of a pole of a Fuchsian function is the same as its order, the conventions as to the order being derived from those for a theta-Fuchsian function by taking the index of the function to be zero.

Now, as at *Proc. Lond. Math. Soc.*, Vol. XXXI., p. 307, let $p+1$ be the least number of arbitrary poles that can be assigned to a Fuchsian function which is to have no others, and let g be the number of irreducible circuits, so that

$$g = n-k+1,$$

where $2n$ is the number of sides of the generating polygon, k the number of cycles of vertices. Then, from what we have just proved,

$$q_1-\nu \nless p,$$

since a Fuchsian function with $q_1-\nu+1$ arbitrary poles and no others can be formed.*

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* It may not be superfluous to point out that, if by some other process it should be possible to construct Fuchsian functions with a lower number of arbitrary poles, so that $p < q_1-\nu$, the vertices of the polygon would still behave like ordinary

14. Other inequalities to be satisfied by the numbers p, q_1, g may be found. Let u be an Abelian integral of the first kind; then $\dfrac{du}{dz}$ is a theta-Fuchsian function of index 1, and is finite everywhere in the polygon and on its boundary. Conversely, if $\theta_1(z)$ is any such theta-Fuchsian function, $\int \theta_1(z)\,dz$ will be a uniform function of z, finite everywhere within the polygon and on its boundary, and the values of this function at corresponding points in different polygons will only differ by multiples of certain moduli of periodicity, the multiples depending only on the particular polygons in question. Thus q_1, the number of theta-Fuchsian functions of index 1 without poles, is the number of Abelian integrals of the first kind, and the argument used (*Proc. Lond. Math. Soc.*, Vol. XXXI., p. 307) shows that this number does not fall below $g-p$. We have then

$$q_1 \nless g-p,$$

and, from § 13,
$$q_1-\nu \nless p,$$

so that
$$2q_1 \nless g+\nu,$$

We must now investigate a superior limit for q_1.

15. The moduli of any two Abelian integrals u, v of the first kind are connected by a well known bilinear relation found by evaluating $\int u\,dv$, taken round the perimeter of the generating polygon, as follows. Let ab, cd be two corresponding sides, so that the expression to be evaluated contains the two terms $\int_a^b u\,dv$, $\int_d^c u\,dv$. Denote the values of u, v at a, b, c, d by u_a, v_a, u_b, ... Then

$u_a-u_e = u_b-u_d$ being a modulus for u,

$v_a-v_c = v_b-v_d$ being the corresponding modulus for v,

$$\int_a^b u\,dv+\int_d^c u\,dv = \int_a^b (u_a-u_c)\,dv = (u_a-u_c)(v_b-v_a) = (u_a-u_c)(v_d-v_c).$$

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points in respect to this process, regard being had to the conventions. For any vertex may be one of the $q_1-\nu+1$ arbitrary poles of the process in the text, and, if the other poles are ordinary points to which the new process applies, the function with $q_1-\nu+1$ poles can be reduced by subtraction of functions formed by the new process, so as to have only $p+1$ poles, one being the vertex in question and the rest arbitrary. The same argument will apply if the pole at the vertex is to be of a higher order, the different orders being treated successively.

The whole integral, being the sum of a set of expressions typified by this, is therefore a homogeneous linear function of the moduli of u. Also the term just written

$$= (u_b - u_d)\, v_b - (u_a - u_c)\, v_a$$

$$= (u_b - u_d)\, v_d - (u_a - u_c)\, v_c$$

$$= u_b\,(v_d - v_b) - u_a\,(v_c - v_a) + u_b v_b - u_a v_a - u_d v_d + u_c v_c.$$

Now the four last terms in this disappear on summation, and the expression sought is, therefore, linear and homogeneous in the moduli of v; in fact the effect of interchanging u, v is simply to change its sign.

Since the modulus of periodicity for an integral of the first or second kind is zero in the case of an elliptic or parabolic substitution,* the contribution of two sides connected by such a substitution to the integral just considered has been taken as zero. This needs no justification if the substitution is elliptic; but, if it is parabolic, there is a difficulty, as the vertex in which the sides meet is an essential singularity of the functions.

This difficulty may, however, be easily avoided. Take the notation used for such a case in §§ 7, 8, 11. We may replace the integral along fc, cg (Fig. § 11) by that along fg. Now u, v are both unchanged by the substitution t, and they are, therefore, both uniform functions of $\exp(-\zeta)$, or, say, Z. The path in the Z-plane corresponding to fg is a closed curve round the origin, and u, v are uniform, finite, and continuous in the domain of the Z-origin, so that $\int u\,dv$, taken round this closed curve in the Z-plane, that is, along fg, will vanish; which was to be proved. A cycle of vertices lying on the circular boundary may be reduced to a single parabolic vertex, and will, therefore, now cause no difficulty.

Thus the value of $\int u\,dv$ round the perimeter of the polygon is a skew-symmetrical bilinear expression in the moduli of periodicity of u, v. This is true when the Abelian integrals u, v are of the first or second kind; but, when both are of the first kind, we have the result that this bilinear expression must vanish, since the uniform function $u\dfrac{dv}{dz}$ has no pole within the contour of integration.

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* *Proc. Lond. Math. Soc.*, Vol. xxxi., pp. 305, 307.

16. It should further be shown that the bilinear relation thus found between the moduli of u, v is not illusory. Now, if there are irreducible circuits on the closed surface into which the polygon is deformed by joining together corresponding sides, there will be at least two pairs of corresponding sides which separate each other; that is, if ab, cd are one pair, and ef, gh the other, the order in which these four sides are met with in going round the perimeter will be, say, ab, ef, dc, hg.[*] Now cut the polygon in two by a line from b to d, and subtract the part $b \ldots ef \ldots d$, adding the corresponding part of the polygon adjoining along hg. Thus, a, b, d, c are four consecutive vertices of the new figure; ab still corresponds to cd; suppose kl to be the side corresponding to bd, the polygon being thus $abdc \ldots h \ldots lk \ldots g \ldots$. Cut this in two by a line from d to l and take off the piece $dc \ldots h \ldots l$, adding on the corresponding part of the polygon adjoining along ab; let m be the vertex of this polygon which corresponds to l; then, in the new polygon, m, b, d, l, k are five consecutive vertices, and mb, ld correspond, as also do bd, kl.

Let S, T be the operations that turn mb into ld, bd into kl, respectively, and let us examine the way in which S, T enter into the relations connecting the fundamental substitutions. The five points m, l, d, b, k belong to the same cycle of vertices, and the sequence $T S^{-1} T^{-1} S$ will occur in the relation arising from this cycle (see *Acta Math.*, Vol. I., pp. 45–7); also the operations S, T will not occur in any other of the relations. Thus the restrictions upon moduli of periodicity in general (*Proc. Lond. Math. Soc.*, Vol. XXXI., p. 305, note) are obeyed if we take arbitrarily the moduli corresponding to S, T and make all the other moduli zero. This will secure the isomorphism referred to in the passage cited last.

Now, if this were the case with the functions u, v,[†] the value of $\int u\,dv$ round the rest of the perimeter would be zero, and from mb and dl we should have the contribution

$$(u_d-u_b)(v_l-v_d),$$

---

[*] It is easily seen that if no two pairs of conjugate sides separate each other, there must be sides adjacent to their conjugates. These pairs being fastened together, the resulting surface must again have sides adjacent to their conjugates. Fastening these together, and continuing the process, we arrive in the end at a simply connected closed surface.

[†] They would not of course be integrals of the first kind: but for the present purpose that does not matter, the object being to show that a relation which must be satisfied when they are of the first kind is not generally satisfied when they are not of the first kind.

from *bd* and *lk*, $(u_l-u_d)(v_k-v_d)$.

The sum of these does not vanish identically, and hence the condition cannot be illusory.

17. If there are other pairs of corresponding sides that separate each other, this reduction may be carried further so as to bring the polygon to a canonical form. Suppose $a'b'$, $c'd'$ to be a pair of corresponding sides, and $e'f'$, $g'h'$ another, the order of the vertices being

$$a'b'\ldots e'f'\ldots mbdlk\ldots d'c'\ldots h'g'\ldots.$$

Cut off the part $k\ldots d'c'$ and add on the corresponding part of the polygon adjoining along $a'b'$; let k' be the vertex of this polygon corresponding to k; then the order of letters in the new perimeter is

$$a'k'\ldots b'\ldots e'f'\ldots mbdlkc'\ldots h'g'\ldots,$$

and $a'k'$, $c'k$ are corresponding sides.

Now let the process of § 16 be used, the sides $a'k'$, $e'f'$, kc', $h'g'$ taking the place of ab, ef, dc, hg. The result will be to bring together two sets of four sides, in each of which the first and third correspond, as also the second and fourth.

If two more pairs of sides separate each other, the same process may be applied again, and so on, until all such pairs are exhausted. Then the perimeter will consist of a series of sets of four sides, in each of which the first corresponds to the third, the second to the fourth, followed, possibly, by a series in which no two pairs of corresponding sides separate each other. If this latter series exists, it must include some pair of corresponding sides adjacent to each other, their common end forming a cycle by itself. Let *abcd*... denote the part of the perimeter now being considered, *ef*, *gf* the first pair of adjacent corresponding sides. Cut off the part *a*...*ef* and add on a corresponding piece of the polygon adjoining along *fg*; then the number of sides is not increased, but the first two have been made to correspond. In the same way the next two may be made to correspond, and so on till every side in this part of the perimeter is adjacent to the corresponding side. The order of the vertices will now be, with changed notation,

$$a_1b_1c_1d_1a_2b_2c_2d_2\ldots,\ a_\kappa b_\kappa c_\kappa d_\kappa e_1f_1e_2f_2\ldots\ e_\rho f_\rho,$$

corresponding sides being

$$a_ib_i,\ c_id_i;\ b_ic_i,\ d_ia_{i+1}\quad(i=1,2,\ldots,\kappa-1);$$

$$a_\kappa b_\kappa,\ c_\kappa d_\kappa;\ b_\kappa c_\kappa,\ d_\kappa e_1;\ e_if_i,\ f_ie_{i+1}\ (i=1,2,\ldots,\rho-1);\ e_\rho f_\rho,\ f_\rho a_1.$$

(Compare Klein, *Math. Annalen*, Vol. XXI., p. 184.)

18. Let S_i, T_i, U_i denote the substitution by which a_ib_i, b_ic_i, e_if_i are transformed into the sides corresponding to them respectively. Then the restrictions on moduli of periodicity for Abelian integrals of the first two kinds are simply that those corresponding to

$$U_1,\ U_2,\ \ldots,\ U_p$$

shall vanish; the others are arbitrary.

Let A_i, B_i be the moduli corresponding to S_i for the functions u, v respectively, and A_i', B_i' those corresponding to T_i. Then the bilinear relation to be satisfied if u, v are everywhere finite is

$$\sum_{i=1}^{i=\kappa}(A_iB_i'-A_i'B_i)=0;$$

this is the well known form.

19. Thus, if any number j of Abelian integrals of the first kind are known, the moduli of periodicity for each of these, as also for any others, must satisfy j linear equations, which are all independent; if they were not, the systems of moduli for the j known functions would not be linearly independent, and, therefore, the j functions themselves would not be. This is made very clear by the form in which the bilinear relation was written at the end of § 18.

Thus the q_1 sets of moduli belonging to the q_1 Abelian integrals of the first kind are common solutions of q_1 linear equations. Since the number of moduli in each set is g, it follows that

$$g-q_1 \nless q_1$$

or

$$2q_1 \ngtr g.$$

But now

$$2q_1 \nless g+\nu,$$

where ν is zero or positive. It follows that $\nu=0$, and

$$q_1=\tfrac{1}{2}g=\tfrac{1}{2}(n-k+1);$$

also $\quad p \ngtr q_1-\nu$ and $\nless g-q_1$, so that $p=q_1$;

these are the desired results.

We have further, when $m>1$,

$$q_m=\tfrac{1}{2}(n-k-1)+(m-1)(n-1)-\Sigma\left\{\frac{m}{\lambda}\right\}.$$

(Compare *Acta Math.*, Vol. I., p. 266.)

20. Lastly, it must be possible to express all theta-Fuchsian functions of index >1 and without poles as theta-Fuchsian series (*Acta Math.*, Vol. I., pp. 244–246, 285). For, if of the q_m such functions of index m not more than q_m-1 can be expressed as series, the coefficients $C_0, C_1, \ldots$ of § 1 must be linear combinations of these q_m-1, and thus by the negative construction a theta-Fuchsian function of index $1-m$ can be formed with q_m assigned poles only. More generally $r+1$ can be formed with q_m+r given poles and no others, and two of these, say θ_{1-m} and θ'_{1-m}, will have $r-1$ assigned zeroes.

Let these $r-1$ zeroes be the poles of any particular function θ_{m-1} of index $m-1$, and assign the zeroes of θ_{m-1} among the q_m+r poles of $\theta_{1-m}, \theta'_{1-m}$. The sum of the orders of the poles thus assigned is

$$r-1+(m-1)\left(n-1-\Sigma\frac{1}{\lambda}\right),$$

and the sum of their ranks is easily found to be

$$r-1+(m-1)(n-1)-\Sigma\left\{\frac{m}{\lambda}\right\}.$$

Thus $$q_m+1-(m-1)(n-1)+\Sigma\left\{\frac{m}{\lambda}\right\} \text{ or } q_1$$

of the poles of $\theta_{1-m}, \theta'_{1-m}$ are still at our disposal. Now we may take the reciprocal of θ_{m-1} to be θ'_{1-m}, and thus the product $\theta_{m-1}\theta_{1-m}$ is not a constant, but is a Fuchsian function having q_1 *arbitrary* poles and no more. This is impossible, since $q_1<p+1$.

Thus every theta-Fuchsian function without poles, of index >1, can be expressed as a theta-Fuchsian series. If a theta-Fuchsian function of index >1 has poles, a theta-Fuchsian series can be formed by the positive construction with the same poles and residues and the same index; the difference between this series and the function will be a theta-Fuchsian function without poles. Hence every theta-Fuchsian function of index >1, with or without poles, can be expressed as a series of Poincaré's form.

Also the product of a theta-Fuchsian and a Fuchsian function, or of two theta-Fuchsian functions, is a theta-Fuchsian function. Thus, any Fuchsian function or theta-Fuchsian function of index 1 can be expressed as the quotient of two theta-Fuchsian series.

[I am indebted to the referees for pointing out some flaws in this article as originally written. The effect of their suggestions has been a considerable increase in its length.]

Sur les déformations finies et sur les systèmes triples de surfaces orthogonales.

Par Gaston Darboux.

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1. Si l'on considère une transformation ponctuelle dans laquelle la relation entre les deux points homologues, de coordonnées x, y, z et X, Y, Z respectivement, est définie par des formules quelconques

$$(1)\quad \begin{cases} X = f(x, y, z), \\ Y = \varphi(x, y, z), \\ Z = \psi(x, y, z), \end{cases}$$

on peut encore envisager cette transformation comme une deformation finie d'un espace continu dans laquelle le point $m(x, y, z)$ est transporté au point $M(X, Y, Z)$. Ce point de vue est évidemment de nature à suggérer des recherches de nature très variée.

Par exemple, à tout élément infiniment petit de courbe qui part de m, la déformation fait correspondre un élément linéaire partant de M, et la relation entre ces deux éléments est évidemment de nature homographique.

Si nous nous proposons de chercher les éléments linéaires partant de m auxquels correspondent des éléments parallèles, nous aurons évidemment à résoudre le système

$$\frac{df}{dx} = \frac{d\phi}{dy} = \frac{d\psi}{dz}.$$

En désignant par λ la valeur commune des rapports précédents et en représentant par les indices 1, 2, 3 les dérivées relatives à x, y, z, nous sommes conduits au système

$$(2)\quad \begin{cases} (f_1 - \lambda)\, dx + f_2\, dy + f_3\, dz = 0, \\ \phi_1\, dx + (\phi_2 - \lambda)\, dy + \phi_3\, dz = 0, \\ \psi_1\, dx + \psi_2\, dy + (\psi_3 - \lambda)\, dz = 0\,; \end{cases}$$

d'où l'on déduit par l'élimination de dx, dy, dz une équation du troisième degré à laquelle devra satisfaire λ. De là résulte le théorème bien connu suivant :—

Dans une transformation ou dans une déformation finie quelconque, il y a en général trois éléments linéaires infiniment petits partant d'un point quelconque qui sont parallèles à leurs homologues.

Ces éléments, qui peuvent être d'ailleurs réels ou imaginaires, forment un trièdre que nous appellerons le *trièdre principal* du point considéré.

2. Dans l'ensemble des déformations finies, proposons-nous de chercher toutes celles pour lesquelles le trièdre principal est tri-rectangle. En nous reportant aux idées de Cauchy et d'Helmholtz, nous dirons que ces déformations sont *dépourvues de rotation.*

Si nous désignons par les caractéristiques d et δ les différentielles relatives à deux éléments linéaires, arêtes du trièdre principal, nous devrons avoir

$$\begin{cases} df = \lambda\, dx, \\ d\phi = \lambda\, dy, \\ d\psi = \lambda\, dz, \end{cases} \qquad \begin{cases} \delta f = \lambda_1 \delta x, \\ \delta\phi = \lambda_1 \delta y, \\ \delta\psi = \lambda_1 \delta z, \end{cases}$$

et aussi, en supposant les axes rectangulaires,

$$dx\,\delta x + dy\,\delta y + dz\,\delta z = 0;$$

de là on déduira l'équation

$$df\,\delta x - \delta f\,dx + d\phi\,dy - d\phi\,\delta y + d\psi\,\delta z - \delta\psi\,dz = 0,$$

que donnera, en développant,

$$(f_2 - \phi_1)(\delta x\,dy - dx\,\delta y) + (\phi_3 - \psi_2)(\delta y\,dz - dy\,\delta z)$$
$$+ (\psi_1 - f_3)(\delta z\,dx - \delta x\,dz) = 0.$$

Les binomes $\delta y\,dz - dy\,\delta z$, ... sont les paramètres directeurs de la droite perpendiculaire aux deux éléments, c'est-à-dire ici de la troisième arête du trièdre principal.

Si donc on désigne par d_1 les différentielles relatives à cette troisième arête, il faudra que l'on ait

$$(\phi_3 - \psi_2)\,d_1 x + (\psi_1 - f_3)\,d_1 y + (f_2 - \phi_1)\,d_1 z = 0.$$

Cette égalité devant avoir lieu pour les trois arêtes du trièdre entraîne nécessairement les suivantes :—

$$\phi_3 = \psi_2, \quad \psi_1 = f_3. \quad f_2 = \phi_1.$$

Ainsi, pour qu'une déformation finie soit dépourvue de rotation, il faut et il suffit que f, ϕ, ψ soient les trois dérivées partielles d'une même fonction.

Par suite, *une déformation sans rotation sera définie par les équations suivantes :—*

$$(3) \quad X = v_1, \quad Y = v_2, \quad Z = v_3,$$

où v_1, v_2, v_3 *sont les dérivées partielles d'une fonction quelconque* v.

Tous ces résultats étant démontrés ou plutôt rappelés, proposons-nous maintenant la question suivante :—

3. Parmi les déformations dépourvues de rotation, en existe-t-il pour lesquelles les faces du trièdre principal relatif à chaque point de l'espace s'ordonnent suivant les plans tangents à trois familles de surfaces, qui seront nécessairement deux-à-deux rectangulaires ?

On peut évidemment résoudre ce problême comme il suit.

On considéra une famille de surfaces (Θ)

$$\theta(x, y, z) = \text{const.},$$

et l'on exprimera que la normale à celle des surfaces qui passe en un point quelconque m est une des arêtes du trièdre principal en m.

Les arêtes de ce trièdre étant définies par les équations déjà données

$$\frac{dv_1}{dx} = \frac{dv_2}{dy} = \frac{dv_3}{dz} = \lambda,$$

il suffira de développer dv_1, dv_2, dv_3 et de remplacer ensuite dx, dy, dz par les quantités θ_1, θ_2, θ_3 proportionnelles aux cosinus-directeurs de la normale à la surface (Θ), ce qui donnera

$$(4) \quad \frac{\delta_\theta v_1}{\theta_1} = \frac{\delta_\theta v_2}{\theta_2} = \frac{\delta_\theta v_3}{\theta_3} = \lambda,$$

le symbole $\delta_\theta f$ étant employé ici, et dans ce qui va suivre, pour exprimer la combinaison suivante :—

$$\delta_\theta f = \theta_1 f_1 + \theta_2 f_2 + \theta_3 f_3 = \delta_f \theta.$$

Si, dans les équations (4), nous faisons, pour le moment, abstraction de λ, nous aurons deux équations aux dérivées partielles du premier ordre auxquelles devra satisfaire la fonction θ. Prenons, par exemple, les suivantes :—

$$(5) \quad \begin{cases} \theta_2 \delta_\theta v_1 - \theta_1 \delta_\theta v_2 = 0, \\ \theta_3 \delta_\theta v_1 - \theta_1 \delta_\theta v_3 = 0. \end{cases}$$

En écrivant qu'elles sont compatibles, c'est-à-dire en écrivant que le crochet de Jacobi relatif à ces deux équations est nul, nous serons d'abord conduits après quelques réductions, à l'équation suivante :—

$$\theta_1\,(\delta_s v_2,\ \delta_s v_3)+\theta_2\,(\delta_s v_3,\ \delta_s v_1)-\theta_3\,(\delta_s v_1,\ \delta_s v_2)=0\,;$$

puis, en replaçant les crochets de Jacobi ( ) par leurs expressions, nous obtiendrons l'équation suivante :—

$$(6)\qquad (\delta_{v_3} v_{12}-\delta_{v_2} v_{13})\,\theta_1^2+(\delta_{v_3} v_{22}-\delta_{v_2} v_{23}+\delta_{v_1} v_{13}-\delta_{v_3} v_{11})\,\theta_1\theta_2+\ldots=0,$$

où les termes non écrits s'obtiennent par des permutations circulaires, et qui est, comme les équations (5), du second ordre par rapport aux dérivées de θ.

Si l'on exclut une solution parasite, les équations (5) fournissent trois systèmes de valeurs seulement pour les rapports mutuels de θ_1, θ_2, θ_3. Si un seul système de ces valeurs vérifiait les équations (6), une seule des trois faces du trièdre principal relatif aux différents points de l'espace serait formée de plans tangents à une famille de surfaces. Nous allons voir que ce cas ne saurait se présenter : Si un système de solutions des équations (5) vérifie l'équation (6), il en est de même des deux autres. En langage géométrique :

Si une des faces du trièdre principal est formée de plans tangents à une famille de surfaces, il en est de même des deux autres faces de ce trièdre principal.

Pour démontrer ce résultat, reportons-nous aux équations (4) et ordonnons-les suivant les dérivées θ. En désignant par les notations $v_{\iota\kappa}$ les dérivées secondes de v, nous aurons les équations

$$(7)\qquad \begin{aligned} (v_{11}-\lambda)\,\theta_1+v_{12}\theta_2+v_{13}\theta_3&=0,\\ v_{21}\theta_1+(v_{22}-\lambda)\,\theta_2+v_{23}\theta_3&=0,\\ v_{31}\theta_1+v_{32}\theta_2+(v_{33}-\lambda)\,\theta_3&=0. \end{aligned}$$

Si l'on désigne par D le déterminant

$$(8)\qquad D=\begin{vmatrix} v_{11}-\lambda & v_{12} & v_{13}\\ v_{21} & v_{22}-\lambda & v_{23}\\ v_{31} & v_{32} & v_{33}-\lambda \end{vmatrix},$$

et par $D_{\iota\kappa}$ le coefficient de l'élément appartenant à la ι-ième ligne et à la κ-ième colonne du déterminant, on déduit, comme on sait, des équations (7), les relations suivantes :—

$$\frac{\theta_1^2}{D_{11}} = \frac{\theta_2^2}{D_{22}} = \dots = \frac{\theta_1 \theta_2}{D_{12}} = \dots,$$

qui permettent de remplacer l'équation (6) par la suivante :—

$$(9) \quad (\delta_{r_3} v_{12} - \delta_{r_2} v_{13})\, D_{11} + \dots + (\delta_{r_3} v_{22} - \delta_{r_2} v_{23} + \delta_{r_1} v_{13} - \delta_{r_3} v_{11})\, D_{12} + \dots = 0;$$

d'où les dérivées θ_1 ont disparu puisque les déterminants D_{11}, D_{12} contiennent seulement λ. Mais λ lui-même disparait comme on s'en assure aisément, de sorte que, si l'on désigne par Δ le déterminant (8) où λ sera égale à zéro, on est conduit à l'unique équation

$$(10) \quad V_{11} \Delta_{11} + \dots + V_{12} \Delta_{12} + \dots = 0,$$

où l'on a posé pour abréger

$$(11) \quad \begin{cases} V_{11} = \delta_{v_3} v_{12} - \delta_{r_2} v_{13}, \\ V_{12} = \delta_{r_3} v_{22} - \delta_{r_2} v_{23} + \delta_{r_1} v_{13} - \delta_{r_3} v_{11}. \end{cases}$$

On a aussi

$$(12) \quad \Delta_{11} = v_{22} v_{33} - v_{23}^2, \quad \Delta_{12} = v_{13} v_{23} - v_{12} v_{23},$$

de sorte qu'il sera très facile de former l'équation (10).

Nous avons en résumé le théorème suivant :—

Si la fonction v satisfait à l'équation aux dérivées partielles (10), *les trois faces du trièdre principal seront tangentes à trois familles de surfaces, nécessairement orthogonales. Si non, aucune des faces du trièdre principal ne demeurera tangente aux surfaces d'une famille quelconque.*

4. L'équation (10) est aux dérivées partielles du troisième ordre ; elle est homogène, ne contient ni la fonction v ni les dérivées premières de cette fonction. De là résulte que, si l'on en connait une solution particulière

$$v = \phi,$$

on en aura une solution plus générale par la formule

$$v = k\phi + ax + by + cz + d,$$

où a, b, c, d, k sont des constantes. Mais il y a plus : les calculs par lesquels λ s'est éliminé de l'équation (10), montrent aussi qu'on pourra ajouter à toute solution connue la combinaison

$$\kappa\,(x^2 + y^2 + z^2),$$

où κ désigne une constante, de sorte qu'à toute solution ϕ on pourra faire correspondre la solution

$$h\phi + \kappa(x^2+y^2+z^2) + ax + by + cz + d,$$

qui contient *six* constantes.

Introduite dans les formules (4) cette propriété, nous donne le théorème suivant :—

Toute transformation dépourvue de rotation peut, sans perdre le caractère précédent, être composée avec une homothétie.

5. Avant de développer et d'appliquer les résultats précédents il importe que je rappelle quelques propositions relatives aux systèmes triples de surfaces orthogonales.

Etant donné un système triple quelconque de surfaces orthogonales, Combescure a montré le premier qu'on peut établir une infinité de transformations ponctuelles de l'espace faisant correspondre au système triple un système triple orthogonal, telles en outre que les plans tangents aux points correspondants des surfaces correspondantes dans les deux systèmes triples soient parallèles. De mon côté j'ai obtenu le théorème de Combescure par une voie nouvelle, et j'ai montré que, si, conservant les notations de Lamé, on représente l'élément linéaire de l'espace par la formule

$$(13) \quad ds^2 = H^2 d\rho^2 + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2,$$

et si l'on forme les trois équations linéaires aux dérivées partielles,

$$(14) \quad \frac{\partial^2 u}{\partial\rho_i \partial\rho} - \frac{1}{H_i}\frac{\partial H_i}{\partial\rho_\kappa}\frac{\partial u}{\partial\rho_i} - \frac{1}{H_\kappa}\frac{\partial H_\kappa}{\partial\rho_i}\frac{\partial u}{\partial\rho_\kappa} = 0 \quad (i, \kappa = 0, 1, 2),$$

auxquelles satisfont les coordonnées x, y, z considérés comme fonctions de ρ, ρ_1, ρ_2, toute solution U des trois équations (14) donnera un des systèmes découverts par Combescure, à l'aide des formules suivantes :—

$$(15) \quad X\frac{\partial x}{\partial\rho_i} + Y\frac{\partial y}{\partial\rho_i} + Z\frac{\partial z}{\partial\rho_i} = \frac{\partial U}{\partial\rho_i} \quad (i = 0, 1, 2) ;$$

d'où l'on déduit presque immédiatement que, si U était exprimée en x, y, z, on aurait aussi

$$(16) \quad X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}.$$

De là résulte qu'*à tout système orthogonal on peut faire correspondre une infinité de ces transformations dépourvues de rotation que nous avons* *considérées.* Il suffit de prendre une solution quelconque des trois équations aux dérivées partielles (14) de l'exprimer au moyen de x, y, z ; la transformation sera alors définie par les formules (16). Comme la solution générale des équations (14) comporte trois fonctions arbitraire d'une seule variable, on voit bien qu'on pourvu obtenir avec le même degré de généralité, une infinité de transformations de la nature de celles que nous étudions.

6. Puisqu'à tout système triple correspondent une infinité de transformations sans rotation, on voit que l'on peut aborder la recherche des systèmes triples en cherchant d'abord de telles transformations, c'est-à-dire en essayant d'intégrer l'équation aux dérivées partielles (10), de sorte que cette équation peut remplacer celle dont j'avais indiquée l'existence dès 1866 dans les *Annales de l'Ecole Normale Supérieure* et qui a été formée en premier lieu par Cayley. Les deux équations sont, l'une et l'autre, du troisième ordre; mais elles ne s'appliquent pas à la même fonction et sont d'ailleurs de formes tout-à-fait différentes.

7. On peut évidemment donner un grand nombre de solutions particulières de l'équation (14) ; je me contenterai de signaler la suivante :—

$$v = ax^m y^n z^p,$$

où a, m, n, p sont des constantes quelconques ; et je remarquerai, en terminant, que l'on exprime la propriété essentielle des déformations précédentes sous la forme suivante :—

On peut, à l'aide de trois familles de surfaces, partager l'espace en petits parallélépipèdes rectangles que la déformation transporte parallèlement à eux-mêmes, en les dilatant ou les contractant dans chacune de leurs trois dimensions.

The Invariant Syzygies of Lowest Degree for any number of Quartics. By A. Young.

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In a previous paper* I found the irreducible system of types for any number of quartics. The reductions were effected by means of systems of equations between forms of the same degree and order; these were written $[abc...f]=R$, &c. If, instead of R, we write the sum of products of forms for which R stands, these systems of equations will give us the syzygies. This method has, as is shown in § 1, the merit of exhaustiveness.

So far as results are concerned, I have only dealt exhaustively with the invariant syzygies up to degree 7. There is no invariant syzygy of degree lower than 7; and all syzygies of this degree are linearly deducible from syzygies of a single form. From this syzygy it is possible to deduce, by means of differential operations, syzygies between covariants and syzygies of higher degree between invariants.

The system of syzygies for two quartics has been worked out by von Gall.† The syzygy degree 7 obtained here is identically zero when fewer than three quartics are under consideration, so that it does not appear in his paper. As some use is made of substitutions, it is convenient to use Roman letters for symbols indicating them.

The following abbreviations are used:—

$\{\mathrm{s}\}$ denotes the sum of powers of the substitution s.

$\{\mathrm{s}_1, \mathrm{s}_2\}$ denotes the sum of the substitutions of the simplest group containing s_1 and s_2; and so on.

$\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \dots\}$ denotes the sum of the substitutions of the symmetric group of the letters $a, b, c, \dots$.

$\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \dots\}'$ denotes the sum of the substitutions of the alternating group, minus the sum of the substitutions which do not belong to the alternating group of the letters $a, b, c, \dots$.

The invariant types in my paper referred to above were written (aba), $(abca)$; they are written here simply (ab), (abc), ...

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* *Proc. Lond. Math. Soc.*, Vol. xxx., p. 290.

† *Math. Ann.*, Bd. xxxiii., p. 197, and Bd. xxxiv., p. 332.

1. All relations between quartic types, including, of course, syzygies, must depend ultimately on the fundamental identities found in § 2 of my previous paper. Hence, if we form from these all possible equations between quartic forms of a given degree and order, we shall obtain all the relations which exist between forms of that degree and order. To construct these equations it is only necessary to take those obtained by multiplying one or other of the identities by a single factor; all other equations being linearly dependent on these. Hence the relations $[abc...g] = R$, &c., already used to obtain the irreducible system of types, must also give all the syzygies.

Having obtained a relation between types of a certain degree, it is possible, by a certain operation, to obtain from it a relation between types of a higher degree. Thus the result of operating with $(kb)\left[k_0\frac{\partial}{\partial b_0}+k_1\frac{\partial}{\partial b_1}+k_2\frac{\partial}{\partial b_2}\right]$ on the symbolical product $(abcd...)$ is $(abkcd...)+(akbcd...)$. Before, however, using this operator, it is necessary to have the relation written out in full in quadratic symbols. Now we shall find it convenient to ignore symbolical factors of the form $[aa]$. When this is done, any single symbolical form represents a single invariant. In fact, the relations as written will be relations between invariants themselves, rather than between quadratic symbolical products. To pass from the actual coefficients of the quartics to quadratic symbols, and *vice versa*, we have the identity

$$(A_0, A_1, A_2, A_3, A_4 \between x_1, x_2)^4 \equiv a_x^2 \equiv (a_0x_1^2+2a_1x_1x_2+a_2x_2^2)^2.$$

Therefore $\quad A_0 = a_0^2, \quad A_1 = a_0a_1, \quad A_3 = a_1a_2, \quad A_4 = a_2^2,$

$$6A_2 = 4a_1^2+2a_0a_2 = 6a_1^2+[aa] = 6a_0a_2-2\,[aa].$$

Hence, if P be the type represented by the symbolical product $(abcd...)$,

$$P = (abcd...)+\Sigma\,[aa]P_a+\Sigma\,[aa][bb]P_{a,b}+\dots.$$

To find P_b, consider the product

$$\begin{aligned}[ab][bc] \equiv B_0a_2c_2-2B_1\,(a_2c_1+a_1c_2)+(B_2+\tfrac{1}{3}\,[bb])(a_2c_0+a_0c_2)\\ +4\,(B_2-\tfrac{1}{6}\,[bb])\,a_1c_1-2B_3\,(a_1c_0+a_0c_1)+B_4a_0c_0\end{aligned}$$

$$= \text{a function of the coefficients } B+\tfrac{1}{3}\,[bb][ac].$$

Therefore $$P_6 = -\tfrac{1}{3}(acd\ldots).$$

The result, then, of operating with $[kb]\left[k_0\frac{\partial}{\partial b_0}+k_1\frac{\partial}{\partial b_1}+k_2\frac{\partial}{\partial b_2}\right]$ on the invariant type $(abcd\ldots)$ is

$$(abkcd\ldots)+(akbcd\ldots)-\tfrac{2}{3}(bk)(acd\ldots).$$

The effect of this operator, it is easy to see, is the same as that of substituting the coefficients of the covariant type $2[bk]\,b_{x^2}k_{x^2}$ for those of the quartic $b_{x^2}^2$.

2. Before proceeding to the discussion of syzygies by the above method, we shall establish certain syzygies which exist for the types expressible as determinants.

First,

$$96\,|\,ABCDE\,|\;|\,FGHKL\,| = \begin{vmatrix} (af) & (ag) & (ah) & (ak) & (al) \\ (bf) & (bg) & (bh) & (bk) & (bl) \\ (cf) & (cg) & (ch) & (ck) & (cl) \\ (df) & (dg) & (dh) & (dk) & (dl) \\ (ef) & (eg) & (eh) & (ek) & (el) \end{vmatrix}.$$

Similarly the product of $|\,ABCDE\,|$ and any form of the type I_6, as also that of two forms I_6, may be expressed as the sum of products of forms I_2 and I_3.

Next, any function homogeneous and linear in the coefficients of each of the five quartics $a_{x^2}^2$, $b_{x^2}^2$, $c_{x^2}^2$, $d_{x^2}^2$, $e_{x^2}^2$ is of the form

$$P = \Sigma\lambda A_{r_1}B_{r_2}C_{r_3}D_{r_4}E_{r_5},$$

where $r = 0, 1, 2, 3, 4$; and can therefore have only five distinct values. Hence

$$\{\mathrm{a, b, c, d, e}\}'P = |\,ABCDE\,|\,Q. \qquad \text{(I.)}$$

Similarly, $$\{\mathrm{a, b, c, d}\}'P = |\,ABCDQ\,|,$$

and, if P be also homogeneous and linear in the coefficients of f_{x^2},

$$\{\mathrm{a, b, c, d, e, f}\}'P = 0. \qquad \text{(II.)}$$

In (I.) put $P = |\,ABCD,\,EF\,|$; the right-hand side must be zero, otherwise Q would be an invariant of the first degree of $f_{x^2}^2$; this gives the identity for the forms $|\,ABCD,\,EF\,|$ (found before).

In (II.) put $P = (fg)\,|ABCDE|$; the result is a relation which may be written

$$(fg)\,|ABCDE| = (ag)\,|FBCDE| + (bg)\,|AFCDE| + (cg)\,|ABFDE| + (dg)\,|ABCFE| + (eg)\,|ABCDF|. \qquad \text{(III.)}$$

In this we may write for F_0, F_1, F_2, F_3, F_4 the coefficients of F_4, $-4F_3$, $6F_2$, $-4F_1$, F_0 in any type $(f\ldots)$, invariant or covariant. The result will be

$$(g\ldots)\,|ABCDE| = \Sigma I_2 J,$$

where J is reducible, except when $(g\ldots)$ belongs to one of the types I_2, I_3. Operate with $[eh]\left[h_0\frac{\partial}{\partial e_0}+h_1\frac{\partial}{\partial e_1}+h_2\frac{\partial}{\partial e_2}\right]$, and we obtain

$$J\,.\,I_6 = \Sigma P_3,$$

where ΣP_3 represents a sum of products of forms, there being at least three forms in each product, and J is any type except I_2 or I_3.

Operate on (III.) with

$$[eh]\left[h_0\frac{\partial}{\partial e_0}+h_1\frac{\partial}{\partial e_1}+h_2\frac{\partial}{\partial e_2}\right][fk]\left[k_0\frac{\partial}{\partial f_0}+k_1\frac{\partial}{\partial f_1}+k_2\frac{\partial}{\partial f_2}\right],$$

$$2\,(gfk)\,|ABCD,\ EH| = 2\,(geh)\,|ABCD,\ FK| + \Sigma P_3.$$

Similarly, $(geh)\,|ABCD,\ FK| = (fkh)\,|ABCD,\ GE| + \Sigma P_3$.

Therefore $(gfk)\,|ABCD,\ EH| = (hfk)\,|ABCD,\ EG| + \Sigma P_3$.

It is not possible to express $I_3 I_6$ in the form ΣP_3; for suppose

$$(abc)\,|DEFG,\ HK| = \Sigma P_3 = \Sigma\left\{I_5 I_2 I_2 + I_4 I_3 I_2 + I_3 I_3 I_3 + I_3 I_2 I_2 I_2\right\}.$$

Then the syzygy would still be true if we supposed the quartics represented by the letters a, b, c, h, k to be all the same—we may without confusion express this supposition by using only one letter for this quartic. Then

$$(aaa)\,|DEFG,\ AA| = \Sigma P_3 = \Sigma\left\{I_5 I_2 I_2 + \ldots\right\}.$$

Operate on this with $\{\mathrm{d, e, f, g}\}'$; a little consideration will show that the result is

$$24\,(aaa)\,|DEFG,\ AA| = \lambda\,(aa)(aa)\,|DEFGA|;$$

and therefore $\qquad (aaa)\,|DEFA,\ AA| = 0.$

Hence $\qquad (aaa) = 0 \quad \text{or} \quad |DEFA, AA| = 0;$

either result being absurd. In exactly the same way, it may be shown that there is no syzygy of the form

$$I_3 I_6 = \Sigma I_4 I_5 + \Sigma P_3$$

or of the form $$I_2 I_6 = \Sigma I_3 I_5 + \Sigma P_3.$$

3. For the sake of finding syzygies, the equations of my previous paper must be written out in full. As regards degree 4, there is only one equation,

$$2\,(abcd)+2\,(adbc)+2\,(acdb) = (ab)(cd)+(ad)(bc)+(ac)(db);$$

which is the expanded form of

$$\begin{vmatrix} [ab] & [ac] & [ad] & [aa] \\ [bb] & [bc] & [bd] & [ba] \\ [cb] & [cc] & [cd] & [ca] \\ [db] & [dc] & [dd] & [da] \end{vmatrix} = 0.$$

The only form of equation for degree 5 is

$$[ab\ldots] = (abc)(de)+(abd)(ec)+(abe)(cd)+2\,(ab)(cde). \quad \text{(IV.)}$$

Now it was proved before that

$$60\,|ABCDE| = -\,\{\mathrm{a, b, c, d, e}\}'\,(abcde)+R.$$

In this case $R = \Sigma I_2 I_3$, but $\{\mathrm{a, b, c, d, e}\}'\,\Sigma I_2 I_3 = 0$; hence R must here be zero. It will be convenient to use the form

$$(abcde)' = (abcde)-\tfrac{1}{2}\,\{(ab)(cde)+(bc)(dea)+(cd)(eab)$$
$$+(de)(abc)+(ea)(bcd)\}.$$

Then $$60\,|ABCDE| = -\,\{\mathrm{a, b, c, d, e}\}'\,(abcde)'.$$

Further $$[ab\ldots]' = -\tfrac{1}{12}\,\{\mathrm{a, b, c, d, e}\}\,(ab)(cde).$$

Hence $[ab\ldots]'+[ae\ldots]'-[ac\ldots]'-[ad\ldots]'$

$$= 2\,(abcde)'+2\,(abdce)'-2\,(acbed)'-2\,(acebd)' = 0;$$

and therefore

$$(abdce)'-(acbed)'$$
$$= (acebd)'-(abcde)' = (adecb)'-(acdbe)'$$
$$= \ldots = |ABCDE|$$
$$= \tfrac{1}{3}\,\{(abdce)'+(acebd)'+(acbde)'+(adecb)'+(adcbe)'+(abedc)'\}$$
$$+\tfrac{1}{36}\,\{\mathrm{a, b, c, d, e}\}\,(ab)(cde). \quad \text{(V.)}$$

The equations of degree 5 give nothing in the way of syzygies.

For degree 6 there are two kinds of equation to be discussed—

$$[abc\ldots] = (abcd)(ef)+(abce)(fd)+(abcf)(de)+2\,(abc)(def) \quad \text{(VI.)}$$

and $$[ab,\, cd,\, e,\, f] = (abcd)(ef)+(abef)(cd)+(abfe)(cd)+(ab)(cdef)$$
$$+(ab)(cdfe)+(abe)(cdf)+(abf)(cde)-(ab)(cd)(ef). \quad \text{(VII.)}$$

As in the former problem, we will use, in addition, the equation

$$\begin{aligned} 4\ abc|^2\,|def|^2 &= 2\,[a-b-c-]+\{\mathrm{d,\, e,\, f}\}\,[(ad)(be)(cf)-(ad)(becf) \\ &\qquad -(bd)(aecf)-(cd)(aebf)] \\ &= 4\,(abc)(def) \end{aligned}$$

or $$\begin{aligned} &[a-b-c-] \\ &= 2\,(abc)(def)-\tfrac{1}{2}\,\{\mathrm{d,\, e,\, f}\}\,\big[(ad)(be)(cf)-(ad)(becf)-(bd)(ceaf) \\ &\qquad -(cd)(aebf)\big]. \quad \text{(VIII.)} \end{aligned}$$

Let S represent the sum of the sixty possible forms $(abcdef)$, and operate on (VIII.) with $\{\mathrm{a,\, b,\, c,\, d,\, e,\, f}\}$;

$$\begin{aligned} 72S &= \{\mathrm{a,\, b,\, c,\, d,\, e,\, f}\}\,\big[2\,(abc)(def)-3\,(ad)(be)(cf)+9\,(ad)(becf)\big] \\ &= \{\mathrm{a,\, b,\, c,\, d,\, e,\, f}\}\,\big[2\,(abc)(def)+\tfrac{3}{2}\,(ad)(be)(cf)\big]. \quad \text{(IX.)} \end{aligned}$$

Now we proved before that

$$\begin{aligned} &6\big[(adbecf)+(afbdce)+(aebfcd)\big] \\ &= \{\mathrm{(abc),\, (def)}\}\,\big[[ad,\, be,\, c,\, f]+\tfrac{1}{3}\,[abc\ldots]-\tfrac{1}{9}S+\tfrac{1}{9}\,[a-b-c-]\big] \\ &= \{\mathrm{(abc),\, (def)}\}\,\big[(cf)(adbe)+(ad)(becf)+(ad)(befc)+(be)(adcf) \\ &\qquad +(be)(adfc)-(ad)(be)(cf)+(adc)(bef)+(adf)(bec) \\ &\qquad +(ef)(abcd)+\tfrac{2}{3}(abc)(def)-\tfrac{2}{9}\,(abc)(def) \\ &\qquad -2\,(abd)(cef)-\tfrac{1}{3}\,(ad)(be)(cf)-\tfrac{1}{3}\,(ad)(bf)(ce) \\ &\qquad -(ab)(de)(cf)+\tfrac{2}{9}\,(abc)(def)-\tfrac{1}{6}\,(ad)(be)(cf) \\ &\qquad -\tfrac{1}{6}\,(ad)(bf)(ce)+(ad)(becf)\big] \\ &= \{\mathrm{(abc),\, (def)}\}\,\big[\tfrac{2}{3}\,(abc)(def)+(ad)\,\{4\,(becf)+2\,(befc)\} \\ &\qquad -\tfrac{1}{2}\,(ab)(de)(cf)-\tfrac{1}{2}\,(ad)(bf)(ce) \\ &\qquad -\tfrac{3}{2}\,(ad)(be)(cf)\big]. \quad \text{(X.)} \end{aligned}$$

We proceed to find the exact expression for $(abcdef)$ in terms of the determinant forms degree 6 and products of types of lower degree. For this purpose, it is useful to note that $(abcdef)$ is unchanged by any substitution of the group $\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}$. Operate, then, with the sum of the substitutions of this group on the expression given by (X.) for

$$(abcfed)+(afcdeb)+(adcbef).$$

The result is

$$\begin{aligned}\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[3\,(abcfed)-(ace)(bdf)-(ab)\,\{2\,(cfed)+(cedf)\}\\ -(ad)\,\{2\,(cbef)+(cefb)\}-(af)\,\{2\,(cdeb)+(cebd)\}\\ +\tfrac{1}{4}\,(ab)(ce)(df)+\tfrac{1}{4}\,(ad)(ce)(bf)+\tfrac{1}{4}\,(af)(ce)(bd)\\ +\tfrac{1}{4}\,(ab)(cd)(ef)+\tfrac{1}{4}\,(ad)(cf)(eb)+\tfrac{1}{4}\,(af)(cb)(ed)\\ +\tfrac{3}{4}\,(ab)(cf)(de)+\tfrac{3}{4}\,(ad)(cb)(ef)+\tfrac{3}{4}\,(af)(cd)(eb)\big]=0\end{aligned}$$

or

$$\begin{aligned}\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[3\,(abcfed)-(ace)(bdf)\\ -2\,(ab)\,\{2\,(cdef)+(cedf)\}-(ad)\,\{2\,(bcfe)+(bcef)\}\\ +\tfrac{1}{2}\,(ab)(ce)(df)+\tfrac{1}{2}\,(ab)(cd)(ef)+\tfrac{1}{4}\,(ad)(ce)(bf)\\ +\tfrac{1}{4}\,(ad)(cf)(be)+\tfrac{9}{4}\,(ab)(cf)(de)\big]=0. \qquad \textbf{(XI.)}\end{aligned}$$

Similarly, by operating on (X.) itself,

$$\begin{aligned}\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[2\,(abecfd)+(abfdec)-(abc)(def)\\ -\tfrac{1}{3}\,(ab)\,\{4\,(cedf)+2\,(cefd)\}-\tfrac{1}{3}\,(ad)\,\{5\,(becf)+4\,(befc)\}\\ +\tfrac{1}{6}\,(ab)(cd)(ef)+\tfrac{5}{12}\,(ab)(de)(cf)+\tfrac{1}{2}\,(ab)(ce)(df)\\ -\tfrac{1}{12}\,(ad)(ce)(bf)+\tfrac{3}{4}\,(ad)(be)(cf)\big]=0. \qquad \text{(XII.)}\end{aligned}$$

Operate on equations (V.) with $[fc]\left[f_0\dfrac{\partial}{\partial c_0}+f_1\dfrac{\partial}{\partial c_1}+f_2\dfrac{\partial}{\partial c_2}\right]$; then

$$\begin{aligned}|\,DEAB, CF\,|\\ =(acfebd)+(afcebd)-(abcfde)-(abfcde)\\ -\tfrac{2}{3}\,(cf)\,\{(aebd)-(abde)\}-(acf)(ebd)-(cfe)(bda)+(bcf)(dea)\\ +(cfd)(eab)-\tfrac{1}{2}\,(eb)\,\{(dacf)+(dafc)\}-\tfrac{1}{2}\,(bd)\,\{(acfe)+(afce)\}\\ -\tfrac{1}{2}\,(da)\,\{(cfeb)+(fceb)\}+\tfrac{1}{2}\,(de)\,\{(abcf)+(abfc)\}\\ +\tfrac{1}{2}\,(ea)\,\{(bcfd)+(bfcd)\}+\tfrac{1}{2}\,(ab)\,\{(cfde)+(fcde)\}\\ +\tfrac{1}{3}\,(cf)\,\{2\,(eb)(da)-2\,(de)(ab)\}\end{aligned}$$

$$\begin{aligned}
&= (adecfb)+(adefcb)-(acfdbe)-(afcdbe)-\tfrac{2}{3}(cf)\{(adeb)-(adbe)\}\\
&\quad -(ecf)(bad)-(cfb)(ade)+(acf)(dbe)+(cfd)(bea)\\
&\quad -\tfrac{1}{2}(ba)\{(decf)+(defc)\}-\tfrac{1}{2}(ad)\{(ecfb)+(efcb)\}\\
&\quad -\tfrac{1}{2}(de)\{(cfba)+(fcba)\}+\tfrac{1}{2}(db)\{(eacf)+(eafc)\}\\
&\quad +\tfrac{1}{2}(be)\{(acfd)+(afcd)\}+\tfrac{1}{2}(ea)\{(cfdb)+(fcdb)\}\\
&\quad +\tfrac{1}{3}(cf)\{2(ab)(de)-2(bd)(ea)\}.
\end{aligned}$$

Operate on these with $\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}$;

$$\begin{aligned}
&\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,|DEAB, CF|\\
&= \{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[2(abecfd)-2(abcefd)\\
&\qquad +(ab)\{(cdef)+(cedf)\}-\tfrac{1}{3}(ad)\{5(becf)+3(befc)-2(bcef)\}\\
&\qquad +\tfrac{2}{3}(ad)(be)(cf)-\tfrac{2}{3}(ab)(de)(cf)\big]\\
&= \{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[(abcfed)-(acfdbe)\\
&\qquad -(ab)\{(cdef)+(cedf)\}+(ac)\{(bedf)+(befd)\}\\
&\qquad -\tfrac{2}{3}(ad)\{(befc)-(becf)\}+\tfrac{2}{3}(ab)(de)(cf)-\tfrac{2}{3}(ad)(ce)(bf)\big].
\end{aligned}$$

Hence, using $[ad, be, c, f] = R$,

the last expression for $\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,|DEAB, CF|$ becomes

$$\begin{aligned}
&\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[2(abcfed)+4(abecfd)-2(abd)(cef)\\
&\qquad -(ab)\{(cdef)+(cedf)\}+(ac)\{(bedf)+(befd)\}\\
&\qquad -\tfrac{1}{3}(ad)\{7(becf)+8(befc)\}+\tfrac{2}{3}(ab)(de)(cf)\\
&\qquad -\tfrac{2}{3}(ad)(ce)(bf)+(ad)(be)(cf)\big]\\
&= \{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[4(abecfd)+\tfrac{2}{3}(ace)(bdf)-2(abd)(cef)\\
&\qquad +\tfrac{1}{3}(ab)\{5(cdef)+(cedf)\}+(ac)\{(bedf)+(befd)\}\\
&\qquad -\tfrac{1}{3}(ad)\{7(becf)+4(befc)-2(bcef)\}-\tfrac{1}{3}(ab)(ce)(df)\\
&\qquad -\tfrac{1}{3}(ab)(cd)(ef)-\tfrac{5}{6}(ab)(de)(cf)-\tfrac{5}{6}(ad)(ce)(bf)\\
&\qquad +\tfrac{5}{6}(ad)(be)(cf)\big],
\end{aligned}$$

using (XI.).

From $[ab, de, c, f]$, we have

$$\begin{aligned}
&\{(\mathrm{abcdef}), (\mathrm{ac})(\mathrm{df})\}\,\big[(abcdef)+4(abcefd)+(abfdec)-2(abc)(def)\\
&\qquad -2(ab)\{(decf)+(defc)\}-(ad)(bcef)+(ab)(de)(cf)\big] = 0.
\end{aligned}$$

Eliminate $(abfdec)$ between this equation and (XII.); then

$$\begin{aligned}&\{(\mathrm{abcdef}),\ (\mathrm{ac})(\mathrm{df})\}\Big[(abcdef)+4\,(abcefd)-2\,(abecfd)-(abc)(def)\\ &\quad -\tfrac{1}{3}\,(ab)\,\{6\,(cdef)+2\,(cedf)-2\,(cefd)\}\\ &\quad +\tfrac{1}{3}\,(ad)\,\{5\,(becf)+4\,(bcfe)-3\,(bcef)\}-\tfrac{1}{6}\,(ab)(cd)(ef)\\ &\quad +\tfrac{7}{12}\,(ab)(de)(cf)-\tfrac{1}{2}\,(ab)(ce)(df)+\tfrac{1}{12}\,(ad)(ce)(bf)\\ &\quad -\tfrac{3}{4}\,(ad)(be)(cf)\Big]=0.\end{aligned}$$

Eliminate $(abcefd)$ and $(abecfd)$ between this equation and the two expressions we have found for $\{(\mathrm{abcdef}),\ (\mathrm{ac})(\mathrm{df})\}\,|DEAB,\ CF|$; then

$$\begin{aligned}&\tfrac{3}{2}\,\{(\mathrm{abcdef}),\ (\mathrm{ac})(\mathrm{df})\}\,|DEAB,\ CF|\\ &=\{(\mathrm{abcdef}),\ (\mathrm{ac})(\mathrm{df})\}\Big[(abcdef)-(abc)(def)-\tfrac{1}{3}\,(ace)(bdf)\\ &\quad +(abd)(cef)+\tfrac{1}{6}\,(ab)\,\{-5\,(cdef)+7\,(cedf)+4\,(cefd)\}\\ &\quad -\tfrac{1}{2}\,(ac)\,\{(bedf)+(befd)\}+\tfrac{1}{6}\,(ad)\,\{-3\,(becf)\}\\ &\quad -\tfrac{1}{3}(ab)(de)(cf)-\tfrac{1}{3}(ab)(ce)(df)+\tfrac{1}{2}(ad)(ce)(bf)+\tfrac{1}{6}(ad)(be)(cf)\Big].\end{aligned}$$

Therefore

$$\begin{aligned}&2\,(abcdef)\\ &=|ADEB,\ CF|+|EBCF,\ AD|+|CFAD,\ EB|\\ &\quad +\{(\mathrm{abcdef})\}\Big[\tfrac{1}{3}\,(abc)(def)+\tfrac{1}{9}\,(ace)(bdf)-\tfrac{1}{3}\,(abd)(cef)\\ &\quad +\tfrac{1}{2}\,(ab)(cdef)-\tfrac{1}{6}\,(ab)(cedf)-\tfrac{1}{6}\,(ac)(bdef)+\tfrac{1}{6}\,(ad)(becf)\\ &\quad -\tfrac{1}{9}\,(ab)(cd)(ef)+\tfrac{1}{6}\,(ab)(ce)(df)-\tfrac{1}{12}\,(ad)(ce)(bf)\\ &\quad -\tfrac{1}{18}\,(ad)(be)(cf)\Big]. \qquad \text{(XIII.)}\end{aligned}$$

This expression for $(abcdef)$ satisfies the equations $[abc\ldots]=R$ and $[ab,\ cd,\ e,\ f]=R$ identically; hence no syzygies amongst invariants degree 6 can exist.

4. *Degree* 7.—From the invariant identity degree 6, we have

$$\{\mathrm{a,\ b,\ e}\}'\,|FBAD,\ EC|=2\,|EBAD,\ FC|+2\,|FBAE,\ CD|.$$

Hence

$$\begin{aligned}&\{\mathrm{a,\ b,\ e}\}'\big[(afdbec)+(cfdeba)+2\,(efdcba)+(bfdace)\big]\\ &=\tfrac{1}{2}\,\{\mathrm{a,\ b,\ e}\}'\Big[|ABEF,\ DC|+|DCAB,\ EF|+3\,|ADFB,\ CE|\\ &\qquad +|CEAD,\ FB|+|FBCE,\ AD|\\ &\qquad +|ABFC,\ ED|+|EDAB,\ FC|\Big]+R\\ &=6\,|ABEF,DC|+\{\mathrm{a,b,e}\}'\big[(af)(dbec)+(bc)(aefd)-\tfrac{1}{2}(af)(bd)(ec)\big].\end{aligned}$$

Operate on this with $[gf]\left[g_0\frac{\partial}{\partial f_0}+g_1\frac{\partial}{\partial f_1}+g_2\frac{\partial}{\partial f_2}\right]$, and then with $\{\mathrm{g, d, f}\}$; therefore, using the equations $[a\ldots bec]=R$, &c.,

$$6\,\{\mathrm{g, d, f}\}\,|A,\ B,\ E,\ FG,\ DC|$$

$$=\{\mathrm{g, d, f}\}\,\{\mathrm{a, b, e}\}'\left[\tfrac{1}{3}(fg)\,\{2\,(adbec)+(abcde)\}-2\,(afg)(dbec)\right.$$
$$\left.+(afg)(bd)(ec)\right]$$

$$=12\,|A,B,E,FG,DC|+12\,|A,B,E,DF,GC|+12\,|A,B,E,GD,FC|.$$

Hence, operating with $\tfrac{1}{2}\,\{\mathrm{(cd)}\}$,

$$12\,|A,\ B,\ E,\ FG,\ DC|$$

$$=\{\mathrm{(cd),\ (fg)}\}\,\{\mathrm{a, b, e}\}'\left[\tfrac{1}{3}(dg)\,|AFBEC|+(dg)(afbec)\right.$$
$$\left.-2\,(adg)(fbec)+(adg)(ec)(bf)\right],$$

and

$$12\,|A,\ B,\ C,\ E,\ DFG|$$

$$=-2\,\{\mathrm{a, b, c, e}\}'\,|A,\ B,\ E,\ FG,\ DC|$$

$$=\{\mathrm{(fg)}\}\,\{\mathrm{a, b, c, e}\}'\left[\tfrac{1}{12}(dg)\,|AFBEC|+\tfrac{1}{12}(fg)\,|ADBEC|\right.$$
$$\left.-(cg)(afbed)\right].$$

From these we proceed to find an expression for the reducible type $(abcdefg)$.

Operate then on (XIII.) with $[gf]\left[g_0\frac{\partial}{\partial f_0}+g_1\frac{\partial}{\partial f_1}+g_2\frac{\partial}{\partial f_2}\right]$, and we obtain, after multiplying by 12 and reducing,

$$24\,(abcdefg)+24\,(abcdegf)$$

$$=\{\mathrm{(fg),\ (ae)(bd)}\}\left[4\,(fg)(abcde)+6\,\{(ab)(cdefg)+(bc)(defga)\}\right.$$
$$-2\,\{(ab)(cedfg)+(bc)(dfgea)\}$$
$$-\{2\,(ac)(bdefg)+(bd)(acefg)+(ea)(bcdfg)\}+4\,(ad)(becfg)$$
$$+4\,(ace)(bdfg)+4\,(bcd)(aefg)+8\,(abc)(defg)$$
$$-4\,\{(abd)(cefg)+(abe)(cdfg)+(acd)(befg)\}$$
$$+(afg)\,\{6\,(bcde)-2\,(bdce)\}-2\,(bfg)(acde)+2\,(cfg)(adbe)$$
$$+2\,(afg)\,\{(bd)(ce)-2\,(bc)(de)\}$$
$$+2\,(bfg)\,\{(cd)(ae)+(ca)(ed)-(ad)(ce)\}$$
$$-(cfg)\,\{2\,(ad)(be)+(ae)(bd)\}$$

$$-(fg)\{4(ab)(cde)+(ae)(bcd)+(bd)(ace)-2(ac)(bde)\}$$
$$+2(cg)\,|ABDEF|+2(ag)\{(bcdef)-(bdcef)\}$$
$$+2(bg)\{(adcef)+(dfeac)-(afcde)-(adfce)\}$$
$$-(aeg)\{2(bcdf)-2(bdcf)\}+(bdg)\{2(acef)-2(aecf)\}$$
$$-2(abg)\{(fced)-(fcde)\}-2(acg)\{(dbef)-(debf)\}$$
$$-2(bcg)\{(adef)-(daef)\}-(aeg)(bd)(cf)+(bdg)(ae)(cf)$$
$$-2(abg)(cd)(ef)+2(acg)(bd)(ef)+2(abe)(cf)(dg)$$
$$-2(acd)(bf)(eg)\Big].$$

Now. $$24(abcdefg)+24(adcbefg)$$
$$=-24(abdcefg)-24(adbcefg)-24(acbdefg)-24(acdbefg)$$
$$+24(bc)(adefg)+24(cd)(abefg)+24(db)(acefg)+48(bcd)(aefg)$$
$$=\{\mathrm{(bd),\ (ae)(fg)}\}\Big[-2(bd)(acefg)-(ac)\{4(bdefg)-2(bdgef)\}$$
$$+2(fc)\{(abdeg)-2(abdge)\}$$
$$-(ef)\{6(bdcga)+6(bdgac)-4(bdacg)\}-4(fg)(bdcea)$$
$$-4(ae)(bdfcg)+2(eg)\{(bdcfa)+(bdcaf)\}-4(acf)(bdeg)$$
$$-4(cef)(bdag)+4(ace)(bdfg)+4(cfg)(bdae)-16(efg)(bdac)$$
$$+8(aeg)(bdcf)-(abd)\{4(cefg)-2(cfeg)\}$$
$$-(fbd)\{4(aegc)-2(aceg)\}+(cbd)\{6(aefg)+2(aegf)\}$$
$$+2(abd)\{(ce)(fg)-(cg)(ef)\}+(fbd)\{6(ae)(cg)-2(ec)(ag)\}$$
$$+(cbd)\{4(ag)(ef)-2(af)(eg)\}$$
$$+(bd)\{6(ce)(afg)+4(ag)(cef)-2(cf)(aeg)\}$$
$$-4(fd)\,|ABCEG|+(cd)\{10(abefg)+2(abefg)\}$$
$$-2(fd)\{(cbeag)+(abceg)-(abgce)-(ebgac)\}$$
$$-2(ad)\{(ebcfg)+(fbecg)-(cbgef)-(gbfce)\}$$
$$+2(acd)\{(befg)-(begf)\}+2(egd)\{(acbf)-(abcf)\}$$
$$+2(fcd)\{(aegb)-(eagb)\}+2(agd)\{(bcfe)-(bcef)\}$$
$$+2(acd)\{(fb)(eg)+(eb)(fg)\}-2(egd)\{(fb)(ac)+(cb)(af)\}$$
$$-2(fcd)(ab)(eg)+2(agd)(cb)(ef)+2(aef)(cb)(gd)$$
$$-2(agc)(eb)(fd)-2(ace)(fb)(gd)+2(fgc)(ab)(ed)\Big].$$

Similarly, $\qquad -24\,(adcbefg)-24\,(edcbafg)$

$$\begin{aligned}
&= -24\,(ae)(dcbfg)-24\,(fg)\,\{(adcbe)+(edcba)\}\\
&\quad -24\,(bcd)\,\{(efga)+(afge)\}-24\,(dcba)(efg)-24\,(dcbe)(afg)\\
&\quad +24\,(dcb)(ae)(fg)+24\,(dcbeafg)+24\,(dcbaefg)\\
&\quad +24\,(dcbfgea)+24\,(dcbfgae)\\
&= \{(\mathrm{ae}),\,(\mathrm{bd})(\mathrm{fg})\}\,\big[2\,(ae)(dcbfg)\\
&\quad +(cd)\,\{4\,(fgbae)+6\,(bfgae)-2\,(gbfae)\}\\
&\quad +(fg)\,\{-6\,(dcbae)-2\,(dbcae)\}\\
&\quad +(gd)\,\{6\,(fcbae)-2\,(bfcae)-2\,(cbfae)\}-2\,(bd)(fgcae)\\
&\quad +(fd)\,\{4\,(cgbae)-2\,(gcbae)\}+2\,(gc)(bdfae)+4\,(bdf)(cgae)\\
&\quad +4\,(fgd)(cbae)+4\,(gdc)(bfae)-4\,(bcd)(fgae)-8\,(dcf)(gbae)\\
&\quad -4\,(gcf)(bdae)+(bae)\,\{6\,(fgdc)+2\,(fdgc)\}\\
&\quad +(fae)\,\{4\,(dcbg)-2\,(dcgb)\}-2\,(cae)(dbfg)\\
&\quad -(bae)\,\{2\,(fd)(cg)+2\,(fc)(gd)+4\,(fg)(dc)\}\\
&\quad +(fae)\,\{2\,(gb)(cd)-2\,(gd)(cb)\}\\
&\quad +(cae)\,\{2\,(bd)(fg)+2\,(bf)(gd)-2\,(bg)(df)\}\\
&\quad -(ae)\,\{4\,(cb)(fgd)-2\,(fg)(bcd)+2\,(bf)(cdg)+2\,(cg)(bdf)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -2\,(bd)(cfg)-2\,(df)(bcg)\}\\
&\quad +4\,(da)\,|CBEFG|\;+2\,(ca)\,\{(gefbd)-(bedgf)\}\\
&\quad +2\,(ga)\,\{(befcd)+(cebfd)-(fedbc)-(decfb)\}\\
&\quad +2\,(ba)\,\{(cefgd)-(cefdg)\}+4\,(bfa)\,\{(ecgd)-(ecdg)\}\\
&\quad -2\,(bca)\,\{2\,(edfg)-(edgf)-(egdf)\}-(fga)\,\{2\,(edbc)+10\,(ebcd)\}\\
&\quad +2\,(gca)\,\{2\,(efdb)-(efbd)-(ebfd)\}-2\,(bda)\,\{(egcf)-(ecfg)\}\\
&\quad -2\,(dfa)\,\{(ebgc)-(egbc)\}-2\,(bfa)(cg)(de)-2\,(bca)(gd)(fe)\\
&\quad -2\,(fga)(cd)(be)+2\,(gca)(fb)(de)+2\,(bda)(gc)(fe)\\
&\quad +2\,(dfa)(cg)(be)-2\,(dcf)(ba)(ge)-2\,(bdg)(ca)(fe)\\
&\quad +2\,(fbc)(da)(ge)+2\,(fgb)(da)(ce)\big].
\end{aligned}$$

Add these three results, and we obtain, after some reduction,

$$\begin{aligned}
&336\,(abcdefg)\\
&= \{(\mathrm{abcdefg}),\,(\mathrm{bg})(\mathrm{cf})(\mathrm{de})\}\,\big[4\,(ac)\,|BDEFG| + 16\,(ad)\,|BCEFG|\\
&\quad + (ab)\,\{40\,(cdefg) - 8\,(cdfge) - 8\,(cedfg)\}\\
&\quad + (ac)\,\{-8\,(bdefg) - 4\,(debfg) + 24\,(bdgef)\}\\
&\quad + (ad)\,\{-4\,(bcefg) + 16\,(cebfg)\} + (abc)\,\{36\,(defg) - 4\,(efdg)\}\\
&\quad + (abd)\,\{8\,(cfeg) - 16\,(cefg)\} + (abe)\,\{8\,(cfdg) - 24\,(cdfg)\}\\
&\quad + (ace)\,\{16\,(bdfg) - 20\,(bfdg)\}\\
&\quad + (abc)\,\{-4\,(ef)(dg) + 4\,(df)(eg) - 16\,(de)(fg)\}\\
&\quad + (abd)\,\{28\,(ce)(fg) - 12\,(cf)(eg)\}\\
&\quad + (abe)\,\{-2\,(cd)(fg) - 8\,(cg)(df) - 14\,(cf)(dg)\}\\
&\quad + (ace)\,\{-12\,(bd)(\;g) + 2\,(bg)(df) + 6\,(bf)(dg)\}\big].
\end{aligned}$$

Now, if we write

$$\begin{aligned}
(cdefg)'' = (cdefg) + \tfrac{1}{2}\,|CDEFG|&\\
+ \{(\mathrm{cdefg})\}\,\big[-\tfrac{1}{3}(cd)(efg) + \tfrac{1}{6}\,(ce)(dfg)\big]&,
\end{aligned}$$

we shall have $(cdefg)'' = (cegdf)''$

and $\{\mathrm{e},\,\mathrm{f},\,\mathrm{g}\}\,(cdefg) = 0,$

so that the reductions will be much simplified. For the same reason we will use the form

$$(defg)' = (defg) - \tfrac{1}{6}\,\{(de)(fg) + (df)(ge) + (dg)(ef)\}.$$

With this notation we obtain

$$\begin{aligned}
&336\,(abcdefg)\\
&= \{(\mathrm{abcdefg}),\,(\mathrm{bg})(\mathrm{cf})(\mathrm{de})\}\,\big[-20\,(ab)\,|CDEFG| - 2\,(ac)\,|BDEFG|\\
&\quad + 10\,(ad)\,|BCEFG| + (ab)\,\{40\,(cdefg)'' - 8\,(cdfge)'' - 8\,(cedfg)''\}\\
&\quad + (ac)\,\{-8\,(bdefg)'' - 4\,(debfg)'' + 24\,(bdgef)''\}\\
&\quad + (ad)\,\{-4\,(bcefg)'' + 16\,(cebfg)''\}\\
&\quad + (abc)\,\{36\,(defg)' - 4\,(efdg)'\} + (abd)\,\{8\,(cfeg)' - 16\,(cefg)'\}\\
&\quad + (abe)\,\{8\,(cfdg)' - 24\,(cdfg)'\} + (ace)\,\{16\,(bdfg)' - 20\,(bfdg)'\}\\
&\quad + \tfrac{28}{3}\,(abc)\,\{(de)(fg) + (df)(ge) + (dg)(ef)\}\\
&\quad - \tfrac{28}{3}\,(abd)\,\{(ce)(fg) + (cf)(ge) + (cg)(ef)\}\\
&\quad + \tfrac{14}{3}\,(abe)\,\{5\,(cd)(fg) - (cf)(gd) - (cg)(df)\}\\
&\quad + \tfrac{28}{3}\,(ace)\,\{-2\,(bd)(fg) + (bf)(dg) + (bg)(df)\}\big]. \qquad \textbf{(XIV.)}
\end{aligned}$$

If we take this value of $(abcdefg)$ and substitute in the three identities degree 7, viz.,

$$[abcd\ldots] = R, \quad [abc,\, de,\, f,\, g,] = R, \quad [ab,\, cd,\, ef,\, g,] = R,$$

we shall obtain all the syzygies of this degree. Owing to the form of the expression obtained for $(abcdefg)$, it is only necessary to take one equation of each type. Take, then, the equation

$$[abc,\, de,\, f,\, g,] = R\,;$$

this may be written

$$\begin{aligned}
\{(\mathrm{fg}),\, (\mathrm{ac})(\mathrm{de})\}\, \Big[&2\,(abcdefg) + (abcfdeg) + \tfrac{1}{4}\,(fg)\,|ABCDE| \\
&-\tfrac{1}{2}\,(fg)(abcde)'' - (de)(abcfg)'' - (abc)(defg)' - (deg)(abcf)' \\
&-\tfrac{1}{6}\,(abc)\,\{(de)(fg) + 2\,(df)(eg)\} \\
&-\tfrac{1}{12}\,(fg)\,\{4\,(abe)(cd) - 2\,(abd)(ce) + 4\,(ade)(bc) - 2\,(acd)(be) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (bde)(ac)\} \\
&-\tfrac{1}{6}\,(de)\,\{2\,(abg)(cf) + 4\,(afg)(bc) - 2\,(acf)(bg) - (bfg)(ac)\} \\
&-\tfrac{1}{6}\,(deg)\,\{(ab)(cf) + (ac)(fb) + (af)(bc)\}\Big] = 0.
\end{aligned}$$

Hence, putting in the values of the first two forms,

$$\begin{aligned}
\{(\mathrm{fg}),\, (\mathrm{ac})(\mathrm{de})\}\, \Big[&(fg)\,\{-3\,(abcde)'' - (acdbe)'' - 8\,(acbed)''\} \\
&+ (bf)\,\{8\,(cdaeg)'' + 6\,(daceg)'' + 4\,(adecg)''\} \\
&+ (af)\,\{-12\,(ecbdg)'' - 4\,(bdecg)'' + 2\,(debcg)'' + 2\,(ebdcg)''\} \\
&+ (ef)\,\{4\,(cbadg)'' + 6\,(cadbg)'' + 10\,(adcbg)'' - 8\,(bacdg)''\} \\
&- 9\,(ac)(dbefg)'' - 9\,(de)(abcfg)'' + 18\,(cd)(abefg)'' \\
&+ (ab)\,\{8\,(cdefg)'' - 2\,(cedfg)''\} + (eb)\,\{8\,(acdfg)'' - 2\,(cadfg)''\} \\
&- 3\,(bfg)(adce)' + 6\,(afg)\,\{(cdbe)' - (cdeb)'\} \\
&+ (efg)\,\{10\,(abcd)' + 2\,(acdb)'\} - 9\,(acf)(dbeg)' - 9\,(def)(abcg)' \\
&+ 6\,(abf)(cdeg)' + 2\,(ebf)\,\{(cadg)' - (cdag)'\} \\
&+ 4\,(aef)\,\{(cbdg)' - (cdbg)'\} + (adf)\,\{-14\,(bceg)' - 10\,(becg)'\} \\
&+ 6\,(abc)(dfeg)' + 6\,(dbe)(afcg)' - 3\,(abe)(cfdg)' \\
&- 9\,(abd)(cfeg)'\Big] = 0. \qquad \text{(XV.)}
\end{aligned}$$

This may be simplified thus:—Interchange a and e, c and d, and subtract the resulting from the original equation,

$$\begin{aligned}
&\{(\mathrm{fg}),\,(\mathrm{ac})(\mathrm{de})\}\big[(fg)\{-2\,(abcde)''+2\,(acdbe)''\}\\
&\quad+(bf)\{2\,(daceg)''-2\,(adecg)''\}+(af)\{4\,(becdg)''-4\,(debcg)''\}\\
&\quad+(ef)\{4\,(cabdg)''-4\,(badcg)''\}+(afg)\{4\,(cbde)'-4\,(cdeb)'\}\\
&\quad+(efg)\{4\,(dcab)'-4\,(dbca)'\}+(abf)\{4\,(cdeg)'-4\,(cedg)'\}\\
&\quad+(ebf)\{4\,(dacg)'-4\,(dcag)'\}+4\,(aef)\{(dcbg)'-(cdbg)'\}\\
&\quad+(adf)\{4\,(becg)'-4\,(bceg)'\}\big]=0. \qquad \text{(XVI.)}
\end{aligned}$$

Now, let the expression

$$\begin{aligned}
&(gf)(abcde)''+(af)(gbdce)''+(bf)(agced)''+(cf)(adgbe)''\\
&\quad+(df)(abegc)''+(ef)(acbdg)''-(agf)(bced)'-(bgf)(cdae)'\\
&\quad-(cgf)(deba)'-(dgf)(eacb)'-(egf)(abdc)'-(abf)(dcge)'\\
&\quad-(acf)(egdb)'-(adf)(cebg)'-(aef)(gbcd)'-(bcf)(edga)'\\
&\quad-(bdf)(agec)'-(bef)(dacg)'-(cdf)(gbae)'-(cef)(adbg)'\\
&\quad-(def)(gcba)'\equiv\phi\,(abcde,\,g,\,f)\,;
\end{aligned}$$

then $\phi\,(abcde,\,f,\,g)=\phi\,(bcdea,\,f,\,g)=\phi\,(edcba,\,f,\,g)$.

Then equation (XVI.) is equivalent to

$$\phi\,(abcde,\,g,\,f)+\phi\,(abcde,\,f,\,g)=\phi\,(acdbe,\,g,\,f)+\phi\,(acdbe,\,f,\,g).$$

Operate on this equation with the substitutions (bcd), (abcde), $(\mathrm{abcde})^3$, $(\mathrm{abcde})^4$ respectively; and we obtain

$$\begin{aligned}
\phi\,(abcde,\,g,\,f)+\phi\,(abcde,\,f,\,g)&=\phi\,(acdbe,\,g,\,f)+\phi\,(acdbe,\,f,\,g)\\
&=\phi\,(adbce,\,g,\,f)+\phi\,(adbce,\,f,\,g)\\
&=\phi\,(acedb,\,g,\,f)+\phi\,(acedb,\,f,\,g)\\
&=\phi\,(abecd,\,g,\,f)+\phi\,(abecd,\,f,\,g)\\
&=\phi\,(acbed,\,g,\,f)+\phi\,(acbed,\,f,\,g)\\
&=0,
\end{aligned}$$

since the sum of the above six expressions is zero.

Consider then the expression $\phi\,(abcde,\,g,\,f)$; it may be written

$$\begin{aligned}
&\{(\mathrm{abcde})\}\big[\tfrac{1}{5}\,(fg)(abcde)''+(af)(gbdce)''-(agf)(bced)'\\
&\qquad-(abf)(dcge)'-(acf)(egdb)'\big].
\end{aligned}$$

But it might equally well be written

$$\{(\mathrm{gbdce})\}\left[\tfrac{1}{5}(af)(gbdce)''+(gf)(abcde)''-(gaf)(bced)' \right.$$
$$\left. -(gbf)(cdae)'-(gdf)(eacb)'\right].$$

Hence $\phi\,(abcde,\,g,f) = \varphi\,(gbdce,\,a,f) = \varphi\,(agced,\,b,f) = \ldots$.

We have proved that

$$\{(\mathrm{fg})\}\,\phi\,(abcde,\,g,f) = 0\,;$$

hence also

$$\{(\mathrm{fa})\}\,\varphi\,(abcde,\,g,f) = 0, \quad \{(\mathrm{fb})\}\,\varphi\,(abcde,\,g,f) = 0, \quad \&\mathrm{c}.$$

Suppose, now, we express $\phi\,(abcde,\,g,f)$ in linear symbols, the quartics being α_x^4, β_x^4, γ_x^4, δ_x^4, ϵ_x^4, ζ_x^4, η_x^4; then, if ζ and η be interchanged in φ and the result added to the original form, we obtain identically zero. Hence, if we put $\zeta_1/\zeta_2 = \eta_1/\eta_2$ in φ, the result is zero. Similarly, if we give ζ_1/ζ_2 any one of the six values α_1/α_2, β_1/β_2, &c., φ vanishes. Now, φ/ζ_2^4 is a rational integral function of ζ_1/ζ_2, of degree 4, and it vanishes for six values of the variable; hence $\phi \equiv 0$.

The syzygy may be written symmetrically thus

$$\{(\mathrm{abcde}),\,(\mathrm{gbdce})\}\left[(gf)(abcde)''-\tfrac{5}{2}(agf)(bced)'\right] = 0 \quad \text{(XVII.)}$$

or

$$\{(\mathrm{abcde}),\,(\mathrm{gbdce})\}\left[(\dot{g}f)(abcde)-\tfrac{5}{2}(agf)(bced)-\tfrac{5}{3}(gf)(ab)(cde)\right.$$
$$+\tfrac{5}{6}(gf)(ac)(bde)+\tfrac{5}{6}(agf)(bc)(ed)$$
$$\left.+\tfrac{5}{12}(agf)(be)(cd)\right] = 0.$$

All other syzygies degree 7 are reducible to this one. In order to prove this, it is well to further abbreviate the notation. Thus, let

$$(gf)(abcde)_1 \equiv (gf)(abcde)''-\{(\mathrm{abcde})\}\left[\tfrac{1}{2}(agf)(bced)'\right].$$

Then the syzygy just obtained becomes

$$\{(\mathrm{abcde})\}\left[\tfrac{1}{5}(gf)(abcde)_1+(af)(gbdce)_1\right] = 0.$$

The syzygy

$$(fg)\,\{(becda)_1+(bedca)_1\}+(fe)\,\{(gbdac)_1+(gbdca)_1\}$$
$$+(fd)\,\{(gcbea)_1+(gebca)_1\}+(fc)\,\{(egdba)_1+(gedba)_1\}$$
$$+(fa)\,\{(gdecb)_1+(gedcb)_1\}+(eg)\,\{(fbcad)_1+(fbcda)_1\}$$
$$+(gc)\,\{(fdbea)_1+(febda)_1\}+(gd)\,\{(efcba)_1+(fecba)_1\}$$
$$+(ga)\,\{(fcedb)_1+(fecdb)_1\}+(ed)\,\{(gabfc)_1+(gfbac)_1\}$$
$$+(ea)\,\{(gfcbd)_1+(fgcbd)_1\}+(ec)\,\{(gdfab)_1+(gfdab)_1\}$$
$$+(dc)\,\{(gbfae)_1+(gbfea)_1\}+(da)\,\{(gfceb)_1+(gcfeb)_1\}$$
$$+(ac)\,\{(edgfb)_1+(egdfb)_1\} = 0 \qquad \text{(XVIII.)}$$

is also required. It is obtained by subtracting

$$\{\mathrm{(gedca)}\}\left[\tfrac{1}{5}(bf)(gedca)_1+(bg)(fecda)_1\right]$$

from the sum of the syzygies

$$\{\mathrm{(gedca)}\}\left[\tfrac{1}{5}(bf)(gedca)_1+(fg)(becda)_1\right]=0,$$

$$\{\mathrm{(fecda)}\}\left[\tfrac{1}{5}(bg)(fecda)_1+(bf)(gedca)_1\right]=0, \text{ \&c.}$$

First consider equation (XV.). We obtained (XVI.) from it by interchanging a and e, c and d, and subtracting the resulting from the original equation. If we now add, instead of subtracting, and discuss the result, we shall have completely discussed (XV.). Hence we need only consider the result of operating with $\{\mathrm{(ae)(cd)}\}$ on (XV.), *i.e.*,

$$\begin{aligned}\{\mathrm{(fg),\ (ac)(de),\ (ae)(cd)}\}\Big[&(fg)\{-4(bcdea)_1-8(bedac)_1\}\\&+(bf)\{8(cdaeg)_1+10(daceg)_1\}\\&+(af)\{4(bcdeg)_1+8(ebdcg)_1+12(debcg)_1-20(ecbdg)_1\\&\qquad\qquad-4(bdecg)_1\}\\&+(ab)\{16(cdefg)_1-4(cedfg)_1\}-18(ac)(dbefg)_1\\&+18(ae)(cbdfg)_1\Big]=0.\end{aligned}\tag{XIX.}$$

From $\{\mathrm{(cdefg)}\}\left[\tfrac{1}{5}(ab)(cdefg)_1+(bc)(adfeg)_1\right]=0,$

we obtain

$$\{\mathrm{(fg),\ (ac)(de),\ (ae)(cd)}\}\left[4(ab)(cdefg)_1+2(bf)(cdaeg)_1\right]=0.$$

Similarly,

$$\{\mathrm{(fg),\ (ac)(de),\ (ae)(cd)}\}\left[4(ab)(cedfg)_1+2(bf)(ceadg)_1\right]=0.$$

Also
$$\begin{aligned}\{\mathrm{(fg),\ (ac)(de),\ (ae)(cd)}\}\Big[&(bf)(daceg)_1+(fg)(bdcae)_1\\&+(fa)\{2(edbcg)_1+2(bdceg)_1\}\Big]=0\end{aligned}$$

and
$$\begin{aligned}\{\mathrm{(fg),\ (ac)(de),\ (ae)(cd)}\}\Big[&(bf)(ceadg)_1+(fg)(deacb)_1\\&+(fa)\{2(dbegc)_1+2(cedbg)_1\}\Big]=0.\end{aligned}$$

Using these four equations, (XIX.) becomes

$$\begin{aligned}\{\mathrm{(fg),\ (ac)(de),\ (ae)(cd)}\}\Big[&(fg)\{-2(bedca)_1-14(bcdea)_1-8(bedca)_1\}\\&+(af)\{4(ebdcg)_1+4(bcdeg)_1-8(cedbg)_1-8(debcg)_1\\&\qquad\qquad-20(dcebg)_1-20(ecbdg)_1\}\\&-18(ac)(dbefg)_1+18(ae)(cbdfg)_1\Big]=0.\end{aligned}\tag{XX.}$$

But, from (XVIII.),

$$\{(\mathrm{fg}),\ (\mathrm{ac})(\mathrm{de}),\ (\mathrm{ae})(\mathrm{cd})\}\Big[2\,(fg)(bedca)_1-4\,(af)\,\{(becdg)_1+(dbceg)_1\}$$
$$-2\,(ac)(dbefg)_1-2\,(ad)(cbefg)_1+4\,(ae)(cbdfg)_1\Big]=0.$$

Then (XX.) is merely the sum of numerical multiples of this, and two other equations similarly obtained, after interchanging *c* and *e*, and *c* and *a*, respectively in (XVIII.).

Take next the equation

$$[b,\ cd,\ ef,\ ga,] = R;$$

this may be written

$$\{(\mathrm{ce})(\mathrm{df}),\ (\mathrm{ceg})(\mathrm{dfa})\}\Big[6\,(abcdefg)-6\,(ef)(abcdg)-3\,(abg)(cdef)$$
$$+3\,(abg)(cd)(ef)\Big]=0,$$

or, putting in the value of $(abcdefg)$,

$$\{(\mathrm{ce})(\mathrm{df}),\ (\mathrm{ceg})(\mathrm{dfa}),\ (\mathrm{cd})(\mathrm{ef})(\mathrm{ga})\}\Big[(bc)\,\{20\,(defga)_1+16\,(deafg)_1$$
$$+12\,(dfgea)_1\}$$
$$+(cd)\,\{-20\,(befga)_1+8\,(beafg)_1\}+(ce)\,\{24\,(bgfda)_1+12\,(bfdga)_1\}$$
$$+(cf)\,\{4\,(bdega)_1+48\,(begda)_1-8\,(bgdea)_1+4\,(bdgae)_1\}\Big]=0.$$

Eliminate the terms $(bc)(\ldots)_1$, from this by means of (XVII.), and the equation reduces to

$$\{(\mathrm{ce})(\mathrm{df}),\ (\mathrm{ceg})(\mathrm{dfa}),\ (\mathrm{cd})(\mathrm{ef})(\mathrm{ga})\}\Big[20\,(cd)\,\{(begaf)_1-(befga)_1\}$$
$$+20\,(ce)\,\{2\,(begda)_1-(bgdea)_1-(bdgae)_1\}\Big]=0.$$

And this equation is merely the result of operating on (XVIII.) with $\{(\mathrm{ce})(\mathrm{df}),\ (\mathrm{ceg})(\mathrm{dfa}),\ (\mathrm{cd})(\mathrm{ef})(\mathrm{ga})\}$. The equation

$$[b,\ cd,\ ef,\ ga,] = R,$$

then, gives no fresh syzygy.

We may proceed in exactly the same way with the equation

$$[abcd\ldots] = R,$$

with the same result. But the following is better. Suppose the quartics $e_{x^2}^2$, $f_{x^2}^2$, $g_{x^2}^2$, to be all the same; and in the invariant forms denote them all by the same letter *e*. (The form of the equation under consideration is practically equivalent to this supposition.)

Then
$$[abc,\ de,\ e,\ e,] = 2\,(abcdeee)+2\,(abcedee)+2\,(abceede),$$
$$[abc,\ ed,\ e,\ e,] = 2\,(abcedee)+2\,(abceede)+2\,(abceeed),$$
$$[abc,\ ee,\ d,\ e,] = 2\,(abcdeee)+2\,(abceeed)$$
$$+(abcedee)+(abceede);$$

therefore $2\,[abc,\ ee,\ d,\ e_1]+[abc,\ de,\ e,\ e_1]-2\,[abc,\ ed,\ e,\ e_1]$

$$= 6\,(abcdeee) = [abcd\ldots].$$

If the values of these expressions in terms of irreducible types be substituted, the equation is still identically true. Hence the equation

$$[abcd\ldots] = R$$

can give us nothing more than is given by

$$[abc,\ de,\ f,\ g_1] = R.$$

The above proof holds for types of any degree, and so we conclude that this form of equation never requires special consideration.

The expression for $(abcdefg)$ may be simplified with the help of (XVII.). Operating with $\{(\mathrm{abcdefg}),\ (\mathrm{bg})(\mathrm{cf})(\mathrm{de})\}$ on the various syzygies of the form (XVII.), we obtain five independent relations; by means of these (XIV.) reduces to

$$\begin{aligned}
&36\,(abcdefg)\\
&= \{(\mathrm{abcdefg}),\ (\mathrm{bg})(\mathrm{cf})(\mathrm{de})\}\Big[6\,(ab)(cdefg)_1-3\,(ac)(bdefg)_1\\
&\quad -3\,(ad)(cbefg)_1-\tfrac{15}{7}\,(ab)\,|CDEFG|-\tfrac{3}{14}\,(ac)\,|BDEFG|\\
&\quad +\tfrac{15}{14}\,(ad)\,|CBEFG|+(abc)\,\{6\,(defg)'-\tfrac{9}{2}\,(degf)'\}\\
&\quad +(abd)\,\{6\,(cfeg)'-3\,(cefg)'\}-3\,(abe)(cdfg)'\\
&\quad +(ace)\,\{3\,(bdfg)'-\tfrac{3}{2}\,(bfdg)'\}+(abc)\{(de)(fg)+(df)(ge)+(dg)(ef)\}\\
&\quad -(abd)\,\{(ce)(fg)+(cf)(ge)+(cg)(ef)\}\\
&\quad +(abe)\,\{\tfrac{5}{2}\,(cd)(fg)-\tfrac{1}{2}\,(cf)(dg)-\tfrac{1}{2}\,(cg)(df)\}\\
&\quad +(ace)\,\{-2\,(bd)(fg)+(bf)(gd)+(bg)(df)\}\Big]\\
&= \{(\mathrm{abcdefg}),\ (\mathrm{bg})(\mathrm{cf})(\mathrm{de})\}\Big[6\,(ab)(cdefg)-3\,(ac)(bdefg)\\
&\quad -3\,(ad)(cbefg)+6\,(abc)(defg)-6\,(abd)(cefg)-3\,(abe)(cdgf)\\
&\quad -3\,(ace)(bfdg)+(abc)\,\{-(dg)(ef)-4\,(de)(fg)+2\,(df)(eg)\}\\
&\quad +(abd)\,\{4\,(ce)(fg)+4\,(cg)(ef)-2\,(cf)(ge)\}\\
&\quad +(abe)\,\{-(cd)(fg)-(cg)(df)+2\,(cf)(dg)\}\\
&\quad +\tfrac{1}{2}\,(ace)\,\{(bd)(fg)+(bf)(gd)+(bg)(df)\}+\tfrac{6}{7}\,(ab)\,|CDEFG|\\
&\quad -\tfrac{12}{7}\,(ac)\,|BDEFG|+\tfrac{18}{7}\,(ad)\,|BCEFG|\Big].
\end{aligned}$$

5. From (XVII.), syzygies of higher degree may be deduced. Thus, if we operate on (XVII.) with

$$[gh][hk][kl]\left[l_0\frac{\partial}{\partial g_0}+l_1\frac{\partial}{\partial g_1}+l_2\frac{\partial}{\partial g_2}\right],$$

we obtain at once $(fghkl)(abcde) = \Sigma P_3$,

where ΣP_3 is used, as before, to denote a sum of products of forms, there being at least three forms in each product. Again, by means of syzygies deduced from (XVII.) it is possible to show that $I_4 I_5 = \Sigma P_3$ Thus, operate on (XVII.) with $[hg]\left[h_0\frac{\partial}{\partial g_0}+h_1\frac{\partial}{\partial g_1}+h_2\frac{\partial}{\partial g_2}\right]$;

$$2\,(fgh)(abcde)+\{(\mathrm{abcde})\}\big[(af)\{(bdcegh)+(bdcehg)\}+(agfh)(bced)$$
$$-(abf)\{(edcgh)+(edchg)\}-(acf)\{(dbegh)+(dbehg)\}\big]=\Sigma P_3.$$

Operate on this with $\{\mathrm{f, c, d, e}\}'$; then

$$\{\mathrm{f,c,d,e}\}'\,\{\mathrm{g,h}\}\big[(fgh)(abcde)+\tfrac{1}{2}(af)\{|BEGD, CH|+|CHBE,GD|\}$$
$$+\tfrac{1}{2}(bf)\{|CAGE, DH|+|DHCA, GE|\}\big]=\Sigma P_3.$$

Hence

$$\{\mathrm{f, c, d, e}\}'\,\{\mathrm{a, b}\}'\,\{\mathrm{g, h}\}\big[(fgh)(abcde)$$
$$+\tfrac{2}{3}(af)\{|CEGD, BH|+|BECD, GH|\}\big]=\Sigma P_3;$$

and therefore

$$\{\mathrm{f, c, d, e}\}'\,\{\mathrm{a, b}\}'\,\{\mathrm{g, h}\}\big[(fgh)(abcde)+\tfrac{1}{6}(ag)\,|CEFD, BH|$$
$$+\tfrac{1}{3}(agh)\,|BECDF|\big]=\Sigma P_3.$$

Operate now with $[fk]\left[k_0\frac{\partial}{\partial f_0}+k_1\frac{\partial}{\partial f_1}+k_2\frac{\partial}{\partial f_2}\right]$, and we obtain

$$\{\mathrm{c, d, e}\}'\,\{\mathrm{a, b}\}'\,\{\mathrm{g, h}\}\big[-(fgkh)(abcde)$$
$$+(cgh)\{(abfked)+(abkfed)+(abdkfe)+(abdfke)+(abedfk)+(abedkf)\}$$
$$+\tfrac{4}{3}(agh)\,|BECD, FK|\,\big]=\Sigma P_3.$$

Or, since $(cgh)\,|AKEB, DF|-(dgh)\,|AKEB, CF|=\Sigma P_3$,

$$\{\mathrm{c, d, e}\}'\,\{\mathrm{a, b}\}'\,\{\mathrm{f, k}\}\big[-(fgkh)(abcde)+\tfrac{4}{3}(agh)\,|BECD, FK|$$
$$+(cgh)\{|BEDF,KA|+|BFED,KA|+|EDKA,BF|$$
$$+|EKAD,FB|\}\big]=\Sigma P_3;$$

and therefore

$$\{\mathrm{c, d, e}\}'\,\{\mathrm{a, b}\}'\,\{\mathrm{f, k}\}\big[-(fgkh)(abcde)\big]=\Sigma P_3.$$

Hence $(fgkh)\{(abcde)+(abdec)+(abecd)\} = \Sigma P_3$.

Similarly, $(fgkh)\{(abcde)+(cabde)+(bcade)\} = \Sigma P_3$;

therefore $(fgkh)\{(abecd)-(abdce)\} = \Sigma P_3$,

and hence $(fgkh)(abcde) = \Sigma P_3$

Thus the product of any pair of the invariant types I_4, I_5, I_6—with the exception of $I_4 . I_4$—is expressible in the form ΣP_3. But the product of I_2 or I_3 and any other invariant type is not so expressible.

Covariant syzygies may be obtained from (XVII.) by means of combinations of the operations just used with operations of the forms

$$\left[x_2 \frac{\partial}{\partial a_0} - x_1 x_2 \frac{\partial}{\partial a_1} + x_1^2 \frac{\partial}{\partial a_2}\right]^2$$

and $$\left[(-a_0 x_1 x_2 - a_1 x_2^2)\frac{\partial}{\partial b_0} + (a_0 x_1^2 - a_2 x_2^2)\frac{\partial}{\partial b_1} + (a_1 x_1^2 + a_2 x_1 x_2)\frac{\partial}{\partial b_2}\right]^2.$$

The search for covariant syzygies is, however, much simplified by noticing that each of the irreducible types, ${}_2C_4$, ${}_2C_5$, ${}_4C_3$, ${}_4C_4$, ${}_6C_2$, ${}_6C_3$, may be expressed in the form $R+\Sigma J$, where J denotes a Jacobian, and R a sum of products of concomitants; and therefore the product of any pair of these types may be expressed in the form ΣP_3.

On the Differentiation of Single Theta-Functions. *By* Rev. M. M. U. Wilkinson, M.A.

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1. The definition I adopt for the single theta-functions differs somewhat from that which is ordinarily used. Thus, taking the three equations

$$\left.\begin{aligned} \Theta_1(x) &= \Theta(x)\,\mathrm{sn}\,x \\ \Theta_2(x) &= \Theta(x)\,\mathrm{cn}\,x \\ \Theta_3(x) &= \Theta(x)\,\mathrm{dn}\,x \end{aligned}\right\}, \qquad (1)$$

we at once obtain $\Theta \frac{d\Theta_1}{dx} - \Theta_1 \frac{d\Theta}{dx} = \Theta_2 \Theta_3$, and five others, which may be thus represented :

$$\left.\begin{aligned}
M(\Theta\Theta_1) &= (d'-d)\,\Theta\Theta_1 = \Theta_2\Theta_3\\
M(\Theta_2\Theta_1) &= (d'-d)\,\Theta_2\Theta_1 = \Theta_3\Theta\\
M(\Theta_3\Theta_1) &= (d'-d)\,\Theta_3\Theta_1 = \Theta\Theta_2\\
M(\Theta_2\Theta_3) &= (d'-d)\,\Theta_2\Theta_3 = k'^2\Theta\Theta_1\\
M(\Theta_3\Theta) &= (d'-d)\,\Theta_3\Theta = k^2\Theta_2\Theta_1\\
M(\Theta_2\Theta) &= (d'-d)\,\Theta_2\Theta = \Theta_3\Theta_1
\end{aligned}\right\} . \tag{2}$$

The symbol M^n being defined by $M^n(XX')$

$$= (d'-d)^n XX'$$

$$= X\frac{d^nX'}{dx^n} - \ldots + (-1)^r \frac{n!}{r!\,(n-r)!}\,\frac{d^rX}{dx^r}\,\frac{d^{n-r}X'}{dx^{n-r}} + \ldots + (-1)^n \frac{d^nX}{dx^n}X', \tag{3}$$

we further obtain

$$\left.\begin{aligned}
\frac{d^2}{dx^2}(\log\Theta_1 - \log\Theta) &= -\frac{1}{\operatorname{sn}^2 u} + k^2\operatorname{sn}^2 u\\
\frac{d^2}{dx^2}(\log\Theta_2 - \log\Theta) &= -\frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u} + k^2\operatorname{sn}^2 u\\
\frac{d^2}{dx^2}(\log\Theta_3 - \log\Theta) &= -\frac{k^2\operatorname{cn}^2 u}{\operatorname{dn}^2 u} + k^2\operatorname{sn}^2 u
\end{aligned}\right\}, \tag{4}$$

and, taking
$$\frac{d^2}{dx^2}\log\Theta = C - k^2\operatorname{sn}^2 u \tag{5}$$

and integrating twice, determining the constants so that $\Theta(x)$ is an even function and $\Theta 0$ has a constant value conveniently assigned to it, we have Θ, Θ_1, Θ_2, Θ_3 defined.

In Cayley's *Elementary Treatise* C has for value $1-E/K$, and $\Theta(x)$ is a function of x and k^2. In this memoir $\Theta(x)$ is a function of x and of three constants $\Theta 0$, k^2, and C.

We note the equations

$$\left.\begin{aligned}
M^2(\Theta\Theta) &= 2(C\Theta^2 - k^2\Theta_1^2)\\
M^2(\Theta_1\Theta_1) &= 2(C\Theta_1 - \Theta^2)\\
M^2(\Theta_2\Theta_2) &= 2(C\Theta_2^2 - \Theta_3^2)\\
M^2(\Theta_3\Theta_3) &= 2(C\Theta_3^2 - k^2\Theta_2^2)
\end{aligned}\right\} . \tag{6}$$

Taking the first of these, we find, successively,

$$\frac{1}{\Theta}\frac{d^3\Theta}{dx^3}-\frac{3}{\Theta^2}\frac{d\Theta}{dx}\frac{d^2\Theta}{dx^2}+\frac{2}{\Theta^3}\left(\frac{d\Theta}{dx}\right)^3=-2k^2\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x,$$

$$\frac{1}{\Theta}\frac{d^4\Theta}{dx^4}-\frac{4}{\Theta^2}\frac{d\Theta}{dx}\frac{d^3\Theta}{dx^3}+\frac{3}{\Theta^2}\left(\frac{d^2\Theta}{dx^2}\right)^2$$

$$=6\left\{\frac{1}{\Theta}\frac{d^2\Theta}{dx^2}-\frac{1}{\Theta^2}\left(\frac{d\Theta}{dx}\right)^2\right\}^2-2k^2\left(\mathrm{cn}^2x\,\mathrm{dn}^2x-\mathrm{sn}^2x\,\mathrm{dn}^2x-k^2\mathrm{sn}^2x\,\mathrm{cn}^2x\right)$$

$$=6C^2-2k^2-4\left(3C-1-k^2\right)k^2\,\mathrm{sn}^2\,x,$$

and $$M^4(\Theta\Theta)=2\left(6C^2-2k^2\right)\Theta^2-8\left(3C-1-k^2\right)k^2\Theta_1^2, \qquad (7)$$

$$M^4(\Theta\Theta)-4\left(3C-1-k^2\right)M^2(\Theta\Theta)+4\left\{3C^2-2\left(1+k^2\right)C+k^2\right\}\Theta^2=0, \quad (8)$$

Θ, Θ_1, Θ_2, Θ_3 each satisfy this differential equation.

2. If we take

$$X=A\exp\left(\tfrac{1}{2}\mu m^2x^2+2bmx\right)\Theta(mx+\alpha), \qquad (9)$$

so that $$\frac{1}{X}\frac{dX}{dx}=\mu m^2x+2bm+\frac{d}{dx}\log\Theta(mx+\alpha),$$

$$\frac{1}{X}\frac{d^2X}{dx^2}-\frac{1}{X^2}\left(\frac{dX}{dx}\right)^2=\mu m^2+\frac{d^2}{dx^2}\log\Theta(mx+\alpha),$$

we shall obtain, as before,

$$M^4(XX)-4m^2\left\{3(C+\mu)-1-k^2\right\}M^2(XX)$$
$$+4m^4\left\{3(C+\mu)^2-2\left(1+k^2\right)(C+\mu)+k^2\right\}X^2=0, \quad (10)$$

the μ merging into $C+\mu$. So (9) is the complete integral of (10), four fresh constants, A, b, m, α, being introduced.

3. If we require to solve the equation

$$M^4(XX)=0, \qquad (11)$$

$$\left.\begin{aligned}&\text{taking} && k^2=\kappa^2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\\ &\text{and} && 3(C+\mu)=1+k^2\end{aligned}\right\}, \qquad (12)$$

we have $$1-\kappa^2+\kappa^4=0$$

and $$3(C+\mu)^2-2\left(1+k^2\right)(C+\mu)+k^2=0,$$

and $$X=Ae^{2mbx}\Theta(mx+\alpha) \qquad (13)$$

will be the complete integral of (11), C, k^2 being determined by (12).

[(11) admits of three remarkable solutions, viz.,

$$(a) \quad X = x - \frac{A_7 x^7}{7!} + \frac{A_{13} x^{13}}{13!} - \ldots, \tag{14}$$

$$(b) \quad X = 1 - \frac{B_3 x^3}{3!} + \frac{B_6 x^6}{6!} - \ldots, \tag{15}$$

$$(c) \quad X = mx + c.$$

If we call the series (a) $\mathrm{th}_1\, x$, we shall have

$$\mathrm{th}_1\, \kappa^2 x = \kappa^2\, \mathrm{th}_1\, x, \qquad \mathrm{th}_1\, \kappa^4 x = \kappa^4\, \mathrm{th}_1\, x\,; \tag{16}$$

$$\mathrm{th}\, \kappa^2 x = \mathrm{th}_2\, x, \qquad \mathrm{th}\, \kappa^4 x = \mathrm{th}_3\, x\,; \tag{17}$$

th_2, th_3, being what Θ_2, Θ_3, become for the values of C, k^2, given by (12). The further consideration of equation (11) must be postponed, the object of the present memoir being to establish the following propositions.]

4. We have, generally, corresponding to (2),

$$\left.\begin{aligned}
M^{2n+1}(\Theta\Theta_1) &= W(2n+1)\,\Theta_2\Theta_3 \\
M^{2n+1}(\Theta_2\Theta_1) &= W_2(2n+1)\,\Theta_3\Theta \\
M^{2n+1}(\Theta_3\Theta_1) &= W_3(2n+1)\,\Theta\Theta_2 \\
M^{2n+1}(\Theta_2\Theta_3) &= W'(2n+1)\,\Theta\Theta_1 \\
M^{2n+1}(\Theta_3\Theta) &= W_2'(2n+1)\,\Theta_2\Theta_1 \\
M^{2n+1}(\Theta_2\Theta) &= W_3'(2n+1)\,\Theta_3\Theta_1
\end{aligned}\right\}, \tag{18}$$

and

$$\left.\begin{aligned}
M^{2n}(\Theta\Theta_1) &= W(2n)\,\Theta\Theta_1 \\
M^{2n}(\Theta_2\Theta_1) &= W_2(2n)\,\Theta_2\Theta_1 \\
M^{2n}(\Theta_3\Theta_1) &= W_3(2n)\,\Theta_3\Theta_1 \\
M^{2n}(\Theta_2\Theta_3) &= W'(2n)\,\Theta_2\Theta \\
M^{2n}(\Theta_3\Theta) &= W_2'(2n)\,\Theta_3\Theta \\
M^{2n}(\Theta_2\Theta) &= W_3'(2n)\,\Theta_2\Theta
\end{aligned}\right\}. \tag{19}$$

These W functions are all independent of x, being constants, functions of C and k^2.

[Also we have, corresponding to (6),

$$\left.\begin{aligned} M^{2n}(\Theta\Theta) &= U(2n)\,\Theta^2 = 2V(2n)\,\Theta^2 - 2V'(2n)\,\Theta_1^2 \\ M^{2n}(\Theta_1\Theta_1) &= U_1(2n)\,\Theta_1^2 = 2V_1(2n)\,\Theta_1^2 - 2V_1'(2n)\,\Theta^2 \\ M^{2n}(\Theta_2\Theta_2) &= U_2(2n)\,\Theta_2^2 = 2V_2(2n)\,\Theta_2^2 + 2V_2'(2n)\,\Theta_3^2 \\ M^{2n}(\Theta_3\Theta_3) &= U_3(2n)\,\Theta_3 = 2V_3(2n)\,\Theta_3^2 - 2V_3'(2n)\,\Theta_2^2 \end{aligned}\right\}. \qquad (20)$$

$M^{2n+1}(\Theta\Theta)$, &c., all, of course, vanish.

These V functions are, also, all independent of x, and functions of C and k^2, to be determined.]

5. In what follows the symbols Z, Z_1, Z_2, Z_3 and L, L_1, L_2, L_3 are defined as follows:—

$$\left.\begin{aligned} \Theta Z &= \frac{d\Theta}{dx}, & \Theta_1 Z_1 &= \frac{d\Theta_1}{dx} \\ \Theta_2 Z_2 &= \frac{d\Theta_2}{dx}, & \Theta_3 Z_3 &= \frac{d\Theta_3}{dx} \end{aligned}\right\}, \qquad (21)$$

$$\left.\begin{aligned} L_1 &= Z_1 + Z + Z_3 + Z_2 \\ L &= Z_1 + Z - Z_3 - Z_2 \\ L_2 &= Z_1 - Z - Z_3 + Z_2 \\ L_3 &= Z_1 - Z + Z_3 - Z_2 \end{aligned}\right\}; \qquad (22)$$

so that
$$L\frac{dL}{dx} = L_2\frac{dL_2}{dx} = L_3\frac{dL_3}{dx} = -LL_2L_3 \qquad (23)$$

and
$$\frac{dL_1}{dx} + L^2 = 4C, \quad L^2 - L_2^2 = 4, \quad L^2 - L_3^2 = 4k^2, \qquad (24)$$

or we may define L, L_2, L_3 by the equations

$$L = \frac{2}{\operatorname{sn} 2u}, \quad L_2 = \frac{2\operatorname{cn} 2u}{\operatorname{sn} 2u}, \quad L_3 = \frac{2\operatorname{dn} 2u}{\operatorname{sn} 2u}. \qquad (25)$$

[By using L, L_2, L_3, we can pass from one set of formulæ to another very easily, and the expressions obtained will be more elegant.

6. We might proceed by first proving that the V's are constant, and, from that, that the W's are. But, as the following theorem is

true for all functions which admit of successive differentiation, and so is generally true, and holds for all functions, and not theta-functions only, we shall first establish it, and then show that (18) (19), and (20) must be true, since theta-functions are holomorphic, and Θu always finite when u is finite.]

Suppose that, X, X_1 being any functions of x capable of successive differentiation with respect to x,

$$\frac{dX}{dx} = X(Y+\mu x), \quad \frac{dX_1}{dx} = X_1(Y_1+\mu x), \tag{26}$$

and that R_n is generally defined by the equation

$$XX_1 R_n = M^n (XX_1); \tag{27}$$

then we have as follows:—

$$\left.\begin{aligned} R_1 &= Y_1 - Y \\ R_2 + \frac{dR_1}{dx} &= 2\frac{dY_1}{dx} + R_1(Y_1 - Y) + 2\mu \\ R_2 - \frac{dR_1}{dx} &= 2\frac{dY}{dx} + R_1(Y_1 - Y) + 2\mu \\ \dots \quad \dots & \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}\right\}. \tag{28}$$

$$\left.\begin{aligned} R_{n+1} + \frac{dR_n}{dx} &= 2\frac{d^n Y_1}{dx^n} + 2nR_1 \frac{d^{n-1}Y_1}{dx^{n-1}} + \dots \\ &\dots + \frac{2(n)!}{r!(n-r)!} R_r \frac{d^{n-r}Y_1}{dx^{n-r}} + \dots + 2nR_{n-1}\frac{d(Y_1+\mu x)}{dx} + R_n(Y_1 - Y) \\ (-1)^{n+1}\left(R_{n+1} - \frac{dR_n}{dx}\right) &= 2\frac{d^n Y}{dx^n} - 2nR_1\frac{d^{n-1}Y}{dx^{n-1}} + \dots \\ &\quad + (-1)^r \frac{2(n)!}{r!(n-r)!} R_r \frac{d^{n-r}Y}{dx^{n-r}} + \dots \\ &\quad + (-1)^{n-1} R_{n-1} \frac{d(Y+\mu x)}{dx} + R_n(Y_1 - Y) \end{aligned}\right\}. \tag{29}$$

where we shall have

$$\frac{dR_n}{d\mu} = n(n-1)R_{n-2}. \tag{30}$$

7. To establish (29), we proceed as follows :—

$$M^{2n-1}(XX_1) = X\frac{d^{2n+1}X_1}{dx^{2n+1}} - \frac{dX}{dx}\,\frac{d^{2n}X_1}{dx^{2n}}$$

$$-2n\left(\frac{dX}{dx}\,\frac{d^{2n}X_1}{dx^{2n}} - \frac{d^2X}{dx^2}\,\frac{d^{2n-1}X_1}{dx^{2n-1}}\right) + \dots$$

$$\dots + (-1)^r\frac{(2n)!}{r!\,(2n-r)!}\left(\frac{d^rX}{dx^r}\,\frac{d^{2n-r+1}X_1}{dx^{2n-r+1}} - \frac{d^{r+1}X}{dx^{r+1}}\,\frac{d^{2n-r}X_1}{dx^{2n-r}}\right) + \dots$$

$$\dots - 2n\left(\frac{d^{2n-1}X}{dx^{2n-1}}\,\frac{d^2X_1}{dx^2} - \frac{d^{2n}X}{dx^{2n}}\,\frac{dX_1}{dx}\right)$$

$$+\frac{d^{2n}X}{dx^{2n}}\,\frac{dX_1}{dx} - \frac{d^{2n+1}X}{dx^{2n+1}}X_1.$$

In each of these terms we develop the first term by applying Leibnitz's theorem to the second factor, obtaining, for a general term,

$$(-1)^r\frac{(2n)!}{r!\,p!\,(2n-2r-p)!}\,\frac{d^rX}{dx^r}\,\frac{d^pX_1}{dx^p}\,\frac{d^{2n-r-r}(Y_1+\mu x)}{dx^{2n-r-p}};$$

for

$$\frac{d^{2n-r}}{dx^{2n-r}}\{X_1(Y_1+\mu x)\} = \dots + \frac{(2n-r)!}{p!\,(2n-r-p)!}\,\frac{d^pX_1}{dx^p}\,\frac{d^{2n-r-p}(Y_1+\mu x)}{dx^{2n-r-p}} + \dots.$$

In the second term of each of the above terms, we apply Leibnitz's theorem to the first factor, obtaining, for a general term,

$$-(-1)^r\frac{(2n)!}{(2n-r)!\,q!\,(r-q)!}\,\frac{d^qX}{dx^q}\,\frac{d^{r-q}(Y+\mu x)}{dx^{r-q}}\,\frac{d^{2n-r}X_1}{dx^{2n-r}}.$$

Now consider the series

$$XX_1\frac{d^{2n}(Y_1-Y)}{dx^{2n}} + 2n\left(X\frac{dX_1}{dx} - \frac{dX}{dx}X_1\right)\frac{d^{2n-1}(Y_1+Y)}{dx^{2n-1}} + \dots$$

$$\dots + \frac{(2n)!}{(2s)!\,(2n-2s)!}\left\{X\frac{d^{2s}X_1}{dx^{2s}} - 2s\frac{dX}{dx}\,\frac{d^{2s-1}X_1}{dx^{2s-1}} + \dots + \frac{d^{2s}X}{dx^{2s}}X_1\right\}$$

$$\times\frac{d^{2n-2s}(Y_1-Y)}{dx^{2n-2s}}$$

$$+\frac{(2n)!}{(2s+1)!\,(2n-2s-1)!}\left\{X\frac{d^{2s+1}X_1}{dx^{2s+1}} - \dots - \frac{d^{2s+1}X}{dx^{2s+1}}X_1\right\}$$

$$\times\frac{d^{2n-2s-1}(Y_1+Y+2\mu x)}{dx^{2n-2s-1}} + \dots$$

$$\dots + \left\{X\frac{d^{2n}X_1}{dx^{2n}} - \dots + \frac{d^{2n}X}{dx^{2n}}X_1\right\}(Y_1-Y):$$

and the two developments are seen to be identical, and $M^{2n+2}(XX_1)$ can be developed in like manner.

[The identity of the two developments is readily seen by successively developing $M^3(XX_1)$, $M^4(XX_1)$, ... according to the directions.]

8. Hence we have, successively,

$$R_0 = 1,$$

$$R_1 = Y_1 - Y,$$

$$R_2 = \frac{d(Y_1+Y+2\mu x)}{dx} + R_1(Y_1-Y),$$

$$R_3 = \frac{d^2(Y_1-Y)}{dx^2} + 2R_1\frac{d(Y_1+Y+2\mu x)}{dx} + R_2(Y_1-Y),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\left.\begin{aligned} R_{2n} = {} & \frac{d^{2n-1}(Y_1+Y)}{dx^{2n-1}} + (2n-1)R_1\frac{d^{2n-2}(Y_1-Y)}{dx^{2n-2}} + \ldots \\ & \ldots + \frac{(2n-1)!}{(2r+1)!\,(2n-2r-2)!}R_{2r+1}\frac{d^{2n-2r-2}(Y_1-Y)}{dx^{2n-2r-2}} \\ & + \frac{(2n-1)!}{(2r+2)!\,(2n-2r-3)!}R_{2r+2}\frac{d^{2n-2r-3}(Y_1+Y)}{dx^{2n-2r-3}} + \ldots \\ & \ldots + (2n-1)R_{2n-2}\frac{d(Y_1+Y+2\mu x)}{dx} + R_{2n-1}(Y_1-Y) \\ R_{2n+1} = {} & \frac{d^{2n}(Y_1-Y)}{dx^{2n}} + 2nR_1\frac{d^{2n-1}(Y_1+Y)}{dx^{2n-1}} + \ldots \\ & \ldots + \frac{(2n)!}{(2r)!\,(2n-2r)!}R_{2r}\frac{d^{2n-2r}(Y_1-Y)}{dx^{2n-2r}} \\ & + \frac{(2n)!}{(2r+1)!\,(2n-2r-1)!}R_{2r+1}\frac{d^{2n-2r-1}(Y_1+Y)}{dx^{2n-2r-1}} + \ldots \\ & \ldots + 2rR_{2n-1}\frac{d(Y_1+Y+2\mu x)}{dx} + R_{2n}(Y_1-Y) \end{aligned}\right\} . \quad (31)$$

9. Differentiating, we obtain,

$$\frac{dR_1}{dx} = \frac{d(Y_1-Y)}{dx},$$

$$\frac{dR_2}{dx} = \frac{d^2(Y_1+Y)}{dx^2} + 2R_1\frac{d(Y_1-Y)}{dx},$$

and we observe the law

$$\left.\begin{aligned}
\frac{dR_{2r}}{dx} &= \frac{d^{2r}(Y_1+Y)}{dx^{2r}} + 2rR_1\frac{d^{2r-1}(Y_1-Y)}{dx^{2r-1}} + \dots \\
&\quad \dots + \frac{(2r)!}{(2s)!\,(2r-2s)!}R_{2s}\frac{d^{2r-2s}(Y_1+Y)}{dx^{2r-2s}} \\
&\qquad + \frac{(2r)!}{(2s+1)!\,(2r-2s-1)!}R_{2s+1}\frac{d^{2r-2s-1}(Y_1-Y)}{dx^{2r-2s-1}} + \dots \\
&\quad \dots + 2rR_{2r-1}\frac{d(Y_1-Y)}{dx} \\
\frac{dR_{2r+1}}{dx} &= \frac{d^{2r+1}(Y_1-Y)}{dx^{2r+1}} + (2r+1)R_1\frac{d^{2r}(Y_1+Y)}{dx^{2r}} + \dots \\
&\quad \dots + \frac{(2r+1)!}{(2s)!\,(2r-2s+1)!}R_{2s}\frac{d^{2r-2s+1}(Y_1-Y)}{dx^{2r+2s+1}} \\
&\qquad + \frac{(2r+1)!}{(2s+1)!\,(2r-2s)!}R_{2s+1}\frac{d^{2r-2s}(Y_1+Y)}{dx^{2r-2s}} + \dots \\
&\quad \dots + (2r+1)R_{2r}\frac{d(Y_1-Y)}{dx}
\end{aligned}\right\} . \quad (32)$$

Now suppose this law to hold up to $\frac{dR_{2r-1}}{dx}$; then

$$\begin{aligned}
\frac{dR_{2r}}{dx} &= \frac{d^{2r}(Y_1+Y)}{dx^{2r}} + (2r-1)\left\{R_1\frac{d^{2r-1}(Y_1-Y)}{dx^{2r-1}} + \frac{dR_1}{dx}\frac{d^{2r-2}(Y_1-Y)}{dx^{2r-2}}\right\} + \dots \\
&\dots + \frac{(2r-1)!}{(2s+1)!\,(2r-2s-2)!}\left\{R_{2s+1}\frac{d^{2r-2s-1}(Y_1-Y)}{dx^{2r-2s-1}} + \frac{dR_{2s+1}}{dx}\frac{d^{2r-2s-2}(Y_1-Y)}{dx^{2r-2s-2}}\right\} \\
&\quad + \frac{(2r-1)!}{(2s+2)!\,(2r-2s-3)!}\left\{R_{2s+2}\frac{d^{2r-2s-2}(Y_1+Y)}{dx^{2r-2s-2}} + \frac{dR_{2s+2}}{dx}\frac{d^{2r-2s-3}(Y_1+Y)}{dx^{2r-2s-3}}\right\} + \\
&\dots + R_{2r-1}\frac{d(Y_1-Y)}{dx} + \frac{dR_{2r-1}}{dx}(Y_1-Y).
\end{aligned}$$

Substituting in this for ..., $\frac{dR_{2s}}{dx}$, $\frac{dR_{2s+1}}{dx}$, ..., and collecting together all the terms containing $\frac{d^{2r-3s}(Y_1+Y)}{dx^{2r-2s}}$, we obtain

$$\frac{(2r-1)!}{(2s)!\,(2r-2s-1)!}R_2+\frac{(2r-1)!}{(2s-1)!\,(2r-2s)!}R_{2s-1}(Y_1-Y)$$

$$+\frac{(2r-1)!}{1!\,(2r-2)!}\,\frac{(2r-2)!}{(2s-2)!\,(2r-2s)!}R_{2s-2}\frac{d(Y_1+Y+2\mu x)}{dx}+\dots$$

$$\dots+\frac{(2r-1)!}{(2s-1)!\,(2r-2s)!}\,\frac{d^{2s-1}(Y_1+Y)}{dx^{2s-1}}$$

$$=\frac{(2r-1)!}{(2s)!\,(2r-2s-1)!}R_{2s}+\frac{(2r-1)!}{(2s-1)!\,(2r-2s)!}R_{2s}$$

$$=\frac{(2r)!}{(2s)!\,(2r-2s)!}R_{2s},$$

and similarly for the terms containing $\frac{d^{2r-2s-1}(Y_1-Y)}{dx^{2r-2s-1}}$; in like manner, the law can be shown to hold for $\frac{dR_{2r+1}}{dx}$. Thus (32) is established.

[The identity it expresses is readily seen by successively developing $\frac{dR_2}{dx}$, $\frac{dR_3}{dx}$, ... in the manner we here indicate.]

10. The formulæ (29) follow at once from (31) and (32). And (30) can be proved by induction in much the same way. We have, in succession,

$$\frac{dR_1}{d\mu}=0,\quad \frac{dR_2}{d\mu}=2,$$

$$\frac{dR_3}{d\mu}=4R_1+R_1\,\frac{dR_3}{d\mu}=6R_1,$$

and, assuming that, up till $r=n$,

$$\frac{dR_r}{d\mu}=r(r-1)R_{r-2},$$

we have $$\frac{dR_{n+1}}{d\mu}+\frac{d^2R_n}{dx\,d\mu}=\frac{dR_{n+1}}{d\mu}+n\,(n-1)\frac{dR_{n-2}}{dx},$$

$$\frac{dR_{n-1}}{d\mu}+n\,(n-1)\,\frac{dR_{n-2}}{dx}$$

$$=2\,\frac{n\,(n-1)}{1.2}\,\frac{d^{n-2}Y_1}{dx^{n-2}}+\ldots+\frac{2\,(n)!}{r!\,(n-r)!}\,\frac{dR_r}{d\mu}\,\frac{d^{n-r}Y_1}{dx^{n-r}}+\ldots$$

$$+2n\frac{dR_{n-1}}{d\mu}\,\frac{d\,(Y_1+\mu x)}{dx}+2nR_{n-1}+\frac{dR_n}{d\mu}\,(Y_1-Y)$$

$$=n\,(n-1)\left\{2\frac{d^{n-2}Y_1}{dx^{n-2}}+\ldots+\frac{2\,(n-2)!}{(r-2)!\,(n-r)!}\,R_{r-2}\frac{d^{n-2}Y_1}{dx^{n-2}}+\ldots\right.$$

$$\left.+2\,(n-2)\,R_{n-3}\frac{d\,(Y_1+\mu x)}{dx}+R_{n-2}\,(Y_1-Y)\right\}+2nR_{n-1}$$

$$=n\,(n-1)\left(R_{n-1}+\frac{dR_{n-2}}{dx}\right)+2nR_{n-1},$$

so that $$\frac{dR_{n+1}}{d\mu}=(n+1)\,nR_{n-1}.$$

[11. Applying the general formulæ (29) to Θ functions, we observe that the Θ's need not have the same argument, nor the same modulus. In fact, much more general formulæ than (18), (19), and (20) are obtainable.

To obtain (20), put $X=X_1=\Theta\,(x)$,

so that $Y=Y_1=Z-\mu x$;

then we have, in succession,

$$\left.\begin{aligned}
U\,(2)&=2\,\frac{dZ}{dx}=2C-2k^2\,\mathrm{sn}^2\,u\\
U\,(4)&=2\,\frac{d^3Z}{dx^3}+6U\,(2)\,\frac{dZ}{dx}\\
\ldots\quad&\ldots\quad\ldots\quad\ldots\\
U\,(2n)&=2\,\frac{d^{2n-1}Z}{dx^{2n-1}}+\ldots+\frac{2\,(2n-1)!}{(2r)!\,(2n-2r-1)!}\,U\,(2r)\,\frac{d^{2n-2r-1}Z}{du^{2n-2r-1}}+\ldots\\
\ldots\quad&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots
\end{aligned}\right\}\ldots \tag{33}$$

Now, Θx being *holomorphe dans toute l'étendue du plan*, it and its derived functions are never infinite when x is finite. But, since

$$\frac{d^2}{dx^2}\,\mathrm{sn}^r x = r(r-1)\,\mathrm{sn}^{r-2}x - r^2(1+k^2)\,\mathrm{sn}^r x + r(r+1)\,k^2\,\mathrm{sn}^{r+2}x,$$

$U(2n)$ develops into a finite series of even powers of $\mathrm{sn}\,x$. Now $U(2n)\,\Theta^2$ is never infinite when x is finite. So we have

$$U(2n) = 2V(2n) - 2V'(2n)\,\mathrm{sn}^2 x,$$

where $V(2n)$, $V'(2n)$ are constants, functions of C and k^2, and the other equations (20) can be established in like manner.

I find that

$$U(6) = 8\left\{15C^3 - 15k^2C + 2k^2(1+k^2)\right\}$$
$$-8\left\{45C^2 - 30(1+k^2)\,C + 4 + 11k^2 + 4k^4\right\}k^2\,\mathrm{sn}^2 u; \quad (34)$$

or, corresponding to (8), we may write

$$U(6) - 4(3C - 1 - k^2)\,U(4) - 12\left\{3C^2 - 2(1+k^2)\,C + k^2\right\}U(2)$$
$$+96\left\{C^3 - C^2(1+k^2) + Ck^2\right\} = 0. \quad (35)$$

The successive coefficients being symmetrical functions of L, L_2, L_3, the formulæ (8), (35) will be unaltered when Θ_1, Θ_2, Θ_3 take the place of Θ.]

12. In (26), putting

$$X = \Theta(x), \quad X_1 = \Theta_1(x),$$

we have $$Y + \mu x = Z, \quad Y_1 + \mu x = Z_1;$$

then, since $$(Z_1 - Z)\,\Theta\Theta_1 = \Theta_2\Theta_3,$$

we have, in succession,

$$W(1)(Z_1 - Z) = Z_1 - Z,$$

$$W(2) = \frac{d(Z_1+Z)}{dx} + W(1)(Z_1 - Z)^2,$$

$$\frac{dW(2)}{dx} = \frac{d^2(Z_1+Z)}{dx^2} + 2W(1)(Z_1 - Z)\frac{d(Z_1-Z)}{dx},$$

$$W(3)(Z_1 - Z) = \frac{d^2(Z_1-Z)}{dx^2} + 2W(1)(Z_1 - Z)\frac{d(Z_1+Z)}{dx}$$
$$+ W(2)(Z_1 - Z),$$

$$W(3)\frac{d(Z_1-Z)}{dx} = \frac{d^3(Z_1-Z)}{dx^3} + 3W(1)(Z_1 - Z)\frac{d^2(Z_1+Z)}{dx^2}$$
$$+ 3W(2)\frac{d(Z_1-Z)}{dx},$$

$$\left.\begin{aligned}
W(2n) &= \frac{d^{2n-1}(Z_1+Z)}{dx^{2n-1}} + (2n-1)\,W(1)(Z_1-Z)\,\frac{d^{2n-2}(Z_1-Z)}{dx^{2n-2}} + \dots \\
&\quad \dots + \frac{(2n-1)(2n-2)}{1.2}\,W(2n-3)(Z_1-Z)\,\frac{d^2(Z_1-Z)}{dx^2} \\
&\quad + (2n-1)\,W(2n-2)\,\frac{d(Z_1+Z)}{dx} \\
&\quad + W(2n-1)(Z_1-Z)^2 \\
\frac{dW(2n)}{dx} &= \frac{d^{2n}(Z_1+Z)}{dx^{2n}} + 2nW(1)(Z_1-Z)\,\frac{d^{2n-1}(Z_1-Z)}{dx^{2n-1}} + \dots \\
&\quad \dots + \frac{2n(2n-1)(2n-2)}{1.2.3}\,W(2n-3)(Z_1-Z)\,\frac{d^3(Z_1-Z)}{dx^3} \\
&\quad + \frac{2n(2n-1)}{1.2}\,W(2n-2)\,\frac{d^2(Z_1+Z)}{dx^2} \\
&\quad + 2nW(2n-1)(Z_1-Z)\,\frac{d(Z_1-Z)}{dx} \\
(W2n+1)&(Z_1-Z) \\
&= \frac{d^{2n}(Z_1-Z)}{dx^{2n}} + 2nW(1)(Z_1-Z)\,\frac{d^{2n-1}(Z_1+Z)}{dx^{2n-1}} + \dots \\
&\quad \dots + \frac{2n(2n-1)}{1.2}\,W(2n-2)\,\frac{d^2(Z_1-Z)}{dx^2} \\
&\quad + 2nW(2n-1)(Z_1-Z)\,\frac{d(Z_1+Z)}{dx} \\
&\quad + W(2n)(Z_1-Z) \\
\frac{d}{dx}\{W(2n&+1)(Z_1-Z)\} \\
&= \frac{d^{2n+1}(Z_1-Z)}{dx^{2n+1}} + (2n+1)\,W(1)(Z_1-Z)\,\frac{d^{2n}(Z_1+Z)}{dx^{2n}} + \dots \\
&\quad \dots + \frac{(2n+1)\,2n\,(2n-1)}{1.2.3}\,W(2n-2)\,\frac{d^3(Z_1-Z)}{dx^3} \\
&\quad + \frac{(2n+1)\,2n}{1.2}\,W(2n-1)(Z_1-Z)\,\frac{d^2(Z_1+Z)}{dx^2} \\
&\quad + (2n+1)\,W(2n)\,\frac{d(Z_1-Z)}{dx}
\end{aligned}\right\} \quad (36)$$

Here we observe that, since

$$\frac{d(Z_1+Z)}{dx} = -\frac{1}{\operatorname{sn}^2 x} + 2C - k^2 \operatorname{sn}^2 x, \tag{37}$$

$W(2n)$ develops into a finite series of even powers, positive and negative, of $\operatorname{sn} x$. Also, since $W(2n+1)(Z_1-Z)$ develops into such a series multiplied by $\operatorname{sn} x \operatorname{cn} x \operatorname{dn} x$, $W(2n+1)$ does so too.

Now, Θx, $\Theta_1 x$ being holomorphic throughout, and they and their derived functions never infinite for finite values of x, $M^{2n}(\Theta\Theta_1)$, $M^{2n+1}(\Theta\Theta_1)$ are likewise never infinite.

So, if $$W(2n)\,\Theta\Theta_1 = \left(\frac{B'}{\operatorname{sn}^2 x} + C' + D' \operatorname{sn}^2 x\right)\Theta\Theta_1,$$

we must have $$B' = D' = 0.$$

So $W(2n)$ is constant. In like manner, $W(2n+1)$ is constant.

Of course, then, $\dfrac{dW(2n)}{dx}$, $\dfrac{dW(2n+1)}{dx}$ vanish. So, too, the other W's in (18), (19) are all constant.

13. If we express W, W' in terms of the L's, we can at once find W_2, W_3, &c., by permutation.

For small values of n we have as follows :—

$$\left.\begin{array}{ll} W(1) = 1, & W'(1) = k'^2 \\ W_2(1) = 1, & W_2'(1) = k^2 \\ W_3(1) = 1, & W_3'(1) = 1 \end{array}\right\}; \tag{38}$$

$$\left.\begin{array}{l} W(2) = W'(2) = 2C-1-k^2 \\ W_2(2) = W_2'(2) = 2C-k^2 \\ W_3(2) = W_3'(2) = 2C-1 \end{array}\right\}, \tag{39}$$

$$\left.\begin{array}{l} W(3) = 6C-1-k^2 \\ W_2(3) = 6C-4-k^2 \\ W_3(3) = 6C-1-4k^2 \\ W'(3) = k'^2 W(3) \\ W_2'(3) = k^2 W_2(3) \\ W_3'(3) = W_3(3) \end{array}\right\}, \tag{40}$$

$$\left.\begin{aligned} W(4) &= 12C^2-12(1+k^2)C+1+10k^2+k^4 \\ W_2(4) &= 12C^2-12k^2C+k^4 \\ W_3(4) &= 12C^2-12C+1 \end{aligned}\right\}, \tag{41}$$

$$\left.\begin{aligned} W(5) &= 60C^2-20(1+k^2)C+1-6k^2+k^4 \\ &= 16C^2+24C(C-1)+24C(C-k^2)+(C-1)^2 \\ &\qquad -6(C-1)(C-k^2)+(C-k^2)^2 \\ W_2(5) &= 16(C-1)^2+24(C-1)(C-k^2)+24(C-1)C \\ &\qquad +(C-k^2)^2-6(C-k^2)C+C^2 \\ &= 60C^2-20(4+k^2)C+16+24k^2+k^4 \\ W_3(5) &= 60C^2-20(1+4k^2)C+1+24k^2+16k^4 \end{aligned}\right\}. \tag{42}$$

$$\left.\begin{aligned} W(6) &= 120C^3-180(1+k^2)C^2+30(1+10k^2+k^4) \\ &\qquad -1-59k^2-59k^4-k^6 \\ &= 15(2C-1-k^2)^3-14(2C-1-k^2)(1-k^2)^2-32C(1-k^2)^2 \\ W_2(6) &= 15(2C-k^2)^3-14(2C-k^2)k^4-32(C-1)k^4 \\ &= 120C^3-180k^2C^2+30k^4C+32k^4-k^6 \\ W_3(6) &= 120C^3-180C^2+30C-1+32k^2 \end{aligned}\right\}. \tag{43}$$

Extensions of the Riemann-Roch Theorem in Plane Geometry. *By* F. S. Macaulay.

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§ 1. Introduction.

The ordinary form of the theorem known as the Riemann-Roch theorem in the geometry of plane algebraic curves is capable of generalization in more than one respect. In the following paper I have stated and proved what I believe to be the most general form of the theorem (§ 4); and have given a corresponding, but less definite, theorem for a more extended case (§ 6). The paper forms an addition to an earlier one on "The Theorem of Residuation, &c." (*Proc. Lond. Math. Soc.*, Vol. XXXI., pp. 381–423), to which several references are made in the course of the proof (§§ 2–4). Apart, however, from any external reference, I have endeavoured in the first instance to explain what the generalized Riemann-Roch theorem is.

Suppose that two given algebraic curves C_1, C_2, having no common factor, intersect in any manner in the plane, and that the origin is taken at one of their points of intersection. In the most general case C_1, C_2 will have multiple points of any kind at the origin, with any kind of contact. The intersection of C_1 and C_2 at the origin is called a *base-point*, whose *degree* a is the number of ordinary points to which it is equivalent. This base-point a, and the conditions which it supplies for a curve, are determined by finding its a independent equations, which are the linear equations satisfied identically by the coefficients of both C_1P_1 and C_2P_2, where P_1, P_2 are arbitrary power series in x, y.

More generally, the condition that a given system of linear equations may be the equations of a base-point is that the equations are satisfied by the coefficients of CP as a consequence of their being satisfied by the coefficients of C, where, as before, P is arbitrary. Without entering into further details as to the character of the equations, we may add that the particular kind of base-point which is the whole intersection of two given curves at a point is called a *one-set* point, and its set of equations a one-set system.

Base-points include all kinds of points, viz., the ordinary point, of degree 1 and order 1, the equation of which is $z_0^0 = 0$ (z_q^p being the coefficient of $x^{p-q}y^q$ in the general polynomial of any order, so that $p \geqslant q \geqslant 0$); the ordinary i-point, of degree $\frac{1}{2}i(i+1)$ and order i, which supplies an ordinary i-fold point for a curve and no further conditions, whose equations are $z_q^p = 0$ for all values of p, q, such that $i > p \geqslant q$; the one-set point, which is the whole base-point common to any two curves at a point; and, in general, the t-set point, which is the whole base-point common to $t+1$ fixed curves at a point, of any order $i \geqslant t$ and any (finite) degree $q \geqslant \frac{1}{2}i(i+1)$.

A base-point is said to form part of and to be *contained in* another if both are situated at the same point and if the equations of the first are deducible from those of the second. Given any base-point, we can always find an ordinary i-point, and also a one-set point, which contains it.

A *point-base* is any group of base-points (including any number of ordinary points), all finitely separated and *in the finite region of the plane;* and its degree N is the sum of the degrees of its base-points. A point-base forms part of and is contained in another if each of its base-points is contained in a base-point of the other.

If S is the general algebraic curve through any given part q of a one-set point a, then the *remaining part* r of a is that base-point through which S' must pass in order that the composite curve $S+S'$ (analytically SS') may pass through a. The base-points q, r are said to be residual with respect to a, and the sum of their degrees $q+r$ is equal to a.

Similarly, if $N+N'$ is any given point-base *made up of one-set points* (including ordinary points), and N any given part of it, and S the general algebraic curve through N, then the remaining part N' is that point-base through which S' must pass in order that the composite curve $S+S'$ may pass through $N+N'$. The point-bases N, N' are said to be residual with respect to $N+N'$.

With this explanation it is hoped that the enunciation of the generalized Riemann-Roch theorem (§ 4) may be understood. This theorem gives all the defects and excesses of N in terms of those of N when $N+N'$ forms the whole intersection of any two given curves. The n-ic defect of N is the degree of freedom of an n-ic through N; and the n-ic excess of N is the excess of the number N over the number of independent conditions supplied by N for an n-ic.

The generalized Riemann-Roch theorem may be thus contrasted with the ordinary theorem:—The ordinary theorem refers only to the point-base formed by the whole intersection of any given curve C_m with an adjoined curve C_{m-3}; not to the whole intersection of any two given curves. Again, in the ordinary theorem, the point-base N, instead of being *any* part of the whole intersection $N+N'$ of C_m and C_{m-3}, has an ordinary $(i-1)$-point at each multiple point of order i on C_m and $i-1$ on C_{m-3}; the corresponding residual base-point of N' being then also an ordinary $(i-1)$-point. Lastly, the ordinary theorem gives only the values of d'_{m-3} and e'_{m-3}, viz., $d'_{m-3}=e_{m-3}$ and $e'_{m-3}=d_{m-3}$.

§ 2. RESOLUTION OF BASE-POINTS INTO SIMPLE ELEMENTS.

It is easy, in practically endless ways, to resolve any given base-point q into simple elements, that is, to arrange its equations in such a way that any the first q_1 of them shall be the equations of a base-point q_1.[*] It will be sufficient to explain one way of doing this. Let $E = 0$ be any prime equation of the base-point q, and let n be the highest value of p in any z^p_q appearing in E. Arrange the derived equations in the following order:—

$$
\begin{array}{llllll}
(E)^{-n}_{-n} = 0, & & & & & \\
(E)^{-n}_{-n+1} = 0, & (E)^{-n+1}_{-n+1} = 0, & & & & \\
(E)^{-n}_{-n+2} = 0, & (E)^{-n+1}_{-n+2} = 0, & (E)^{-n+2}_{-n+2} = 0, & & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(E)^{-n}_{-0} = 0, & (E)^{-n+1}_{-0} = 0, & (E)^{-n+2}_{-0} = 0, & \ldots, & (E)^{-0}_{-0} = 0. &
\end{array}
$$

Any one of these equations has all its derivatives occurring before it. Thus the equation $(E)^{-l+1}_{-m+1} = 0$ is preceded immediately by its derivative $(E)^{-l}_{-m+1} = 0$, and at a greater distance by its derivative $(E)^{-l}_{-m} = 0$. Hence, if we take the equations in this order, and reject all those which are deducible from equations previously written, the first q_1 which remain are always the equations of a base-point q_1. These equations may be combined in any way with the derivatives of all the other prime equations of q; it being only necessary that the derivatives of the same prime equation preserve the right order among themselves. The reasoning is not affected if two different prime equations have several of their derivatives the same.

In the case of a one-set point $q+r$ a further property can be proved, viz., that its $q+r$ independent equations can be so arranged that the first q of them are the equations of any given base-point q contained in $q+r$, while any the first q_1 of them are the equations of a base-point q_1. If $E = 0$ is the prime equation, and

$$E_1 = E_2 = \ldots = E_{q+r} = 0$$

any set of independent equations of the one-set point, and if S and S' stand for $\Sigma z^p_q x^{p-q} y^q$ and $\Sigma z'^p_q x^{p-q} y^q$ respectively, then the value of E for SS' is identically

$$E_1 F_{q+r} + E_2 F_{q+r-1} + \ldots + E_q F_{r+1} + E_{q+1} F_r + \ldots + E_{q+r} F_1,$$

---

[*] A base-point differs in character from a point-group in respect to the fact that the simple elements from which it is built up must be regarded as having an order of arrangement.

The notation employed is the same as that in the paper of Vol. XXXI. The E's here used have no connexion with the E_n which stands for n-ic excess (§§ 4–6).

to be the equations of the given base-point q; then

$$F_1 = F_2 = \ldots = F_r = 0$$

are the equations of the corresponding residual bas
p. 402). Further, choose $E_1, E_2, \ldots, E_q$ so that
them correspond to a base-point q_1. Also choose
that any the first r_1 of them correspond to a base-p
is effected by a suitable linear transformation of F_1,
corresponding linear transformation of E_{q+1}, E_{q+2},
leaves the value of $E_{q+1}F_r + E_{q+2}F_{r-1} + \ldots + E_{q+r}F$
same as before. Then, since

$$F_1 = F_2 = \ldots = F_{r_1} = 0 \quad (r_1 =$$

are the equations of a base-point r_1,

$$E_1 = E_2 = \ldots = E_{q_1} = 0 \quad (q_1 = q + r - 1,$$

are the equations of the corresponding residual base-
p. 402). Hence any the first q_1 of the equations

$$E_1 = E_2 = \ldots = E_{q+r} = 0$$

are the equations of a base-point q_1.

Also, if S passes through the base-point q_1 who
$E_1 = E_2 = \ldots = E_{q_1} = 0$, and S' through the base-p
equations are $F_1 = F_2 = \ldots = F_{r_1-1} = 0$, where q_1+
SS' passes through a base-point

From this it follows that, if $N+N'$ is any point-base made up of one-set points, and if N is any given part of it, and N' the remaining part, then the conditions which $N+N'$ supplies for a curve may be resolved into simple elements $C_1, C_2, \ldots, C_{N+N'}$,* which can be chosen definitely in such a way that the first N of them are the conditions which the given point-base N supplies for a curve, and any the first N_1 of them the conditions which a point-base N_1 supplies. To these will correspond a unique residual set of conditions $C'_1, C'_2, \ldots, C'_{N+N'}$, such that any the first N'_1 of them are the conditions supplied for a curve by the point base N'_1 which is residual to N_1 with respect to $N+N'$, where $N_1+N'_1 = N+N'$. Moreover, if the conditions $C_1, C_2, \ldots, C_{N_1}$ are satisfied for S, and $C'_1, C'_2, \ldots, C'_{N'_1-1}$ for S', then SS' passes through a point-base $N+N'-1$ contained in $N+N'$. And SS' does not pass through the whole of $N+N'$ unless C_{N_1+1} is satisfied for S, or $C'_{N'_1}$ for S'.

§3. Point-Base forming the whole Intersection of Two Curves.

The $(l+m-3)$-ic excess of the point-base lm formed by the whole intersection of two curves C_l, C_m is 1. In other words, the point-base lm supplies $lm-1$ independent conditions for a curve of order $l+m-3$. It does not necessarily follow that any point-base N forming part of the point-base lm supplies N independent conditions for a curve of order $l+m-3$. Hence the following theorem, which states this to be the fact, is of importance:—

Any curve of order $l+m-3$ through any point-base $lm-1$, forming part of the intersection of two curves C_l, C_m, passes necessarily through the whole intersection of C_l, C_m.

Any point-base $lm-1$ forming part of the point-base lm has a residual point-base of degree 1, which must therefore be an ordinary point. Take the origin at this point. Let C_{l+m-3} be any curve of order $l+m-3$ through the point-base $lm-1$. Then, since x and y are two curves through the residual point-base, we have

$$x\,C_{l+m-3} \equiv C_l A_{m-2} + C_m A_{l-2},$$

and
$$y\,C_{l+m-3} \equiv C_l B_{m-2} + C_m B_{l-2},$$

---

* These conditions are that certain homogeneous linear functions of the coefficients of the curve S, expressed by means of the values of S and the partial differential coefficients of S at the various base-points of $N+N'$, should vanish.

where $A_{l-2}, A_{m-2}, B_{l-2}, B_{m-2}$ are polynomials whose orders are indicated by the suffixes. From these we have

$$x\,(C_l B_{m-2} + C_m B_{l-2}) \equiv y\,(C_l A_{m-2} + C_m A_{l-2}),$$

or

$$C_l\,(xB_{m-2} - yA_{m-2}) \equiv C_m\,(yA_{l-2} - xB_{l-2}).$$

Hence, since C_l, C_m have no common factor, and $xB_{m-2} - yA_{m-2}$ is of less order than C_m, we have

$$xB_{m-2} - yA_{m-2} \equiv 0 \quad \text{and} \quad yA_{l-2} - xB_{l-2} \equiv 0.$$

Hence A_{m-2} and A_{l-2} are both divisible by x; and, writing xA_{m-3}, xA_{l-3} for them, we have

$$C_{l+m-3} \equiv C_l A_{m-3} + C_m A_{l-3},$$

which proves the theorem.

§ 4. THE GENERALIZED RIEMANN-ROCH THEOREM.

If the point-base forming the whole intersection of two curves C_l, C_m, which have no common factor and no intersection at infinity, is divided into any two residual point-bases N, N', and if d_n, d'_n, D_n are the n-ic defects, and e_n, e'_n, E_n the n-ic excesses, of N, N', $N+N'$ respectively, then $d'_{n'} = e_n + D_{n'}$ and $e'_{n'} = d_n - D_n$, where $n + n' = l + m - 3$.

We shall prove first that $e'_{n'} \geqslant d_n - D_n$. Since $d_n + 1$ and $D_n + 1$ are the numbers of linearly independent curves of order n through N and $N+N'$ respectively, it is clear that $d_n - D_n \geqslant 0$. Also, if $d_n - D_n = 0$, then $e'_{n'} \geqslant d_n - D_n$, since $e'_{n'} \geqslant 0$. This shows that the inequality holds in all cases in which N does not lie on an n-ic, for then $d_n = -1 = D_n$, and in all cases in which an n-ic through N passes of necessity through the whole of $N+N'$.

If $d_n - D_n > 0$, we have $d_n \geqslant 0$, since $D_n \geqslant -1$. Also $d_n - D_n$ is the number of (extra) independent conditions which the whole point-base $N+N'$ supplies for an n-ic drawn through N. Hence an n-ic can be drawn through N, and can be made to satisfy $d_n - D_n - 1$ further independent conditions supplied by the point-base $N+N'$, without vanishing identically and without passing through whole of $N+N'$. Let the conditions that $N+N'$ supplies for a curve, viz., $C_1, C_2, \ldots, C_{N+N'}$, be arranged definitely as in § 2; and suppose them applied successively to a curve of order n. Among the conditions $C_{N+1}, C_{N+2}, \ldots, C_{N+N'}$ there will be $d_n - D_n$, each one of which is not a consequence of all the preceding (including $C_1, C_2, \dots, C_N$). Let these be $C_{N_1+1}, C_{N_2+1}, \dots, C_{N_p+1}$, where $N_p > N_{p-1} > \dots > N_1 \geqslant N$ and $p = d_n - D_n$. The last condition C_{N_p+1} will require that the n-ic vanishes identically if $D_n = -1$, otherwise not; but an n-ic can in any case satisfy all the conditions up to C_{N_p}. Now apply the residual set of conditions $C'_1, C'_2, \dots, C'_{N'}$ to a curve of order n'. Then the condition $C'_{N'_p}$, where $N_p + N'_p = N + N'$, will be satisfied as a consequence of the conditions $C'_1, C'_2, \dots, C'_{N'_p-1}$. For an n-ic which satisfies the conditions $C_1, C_2, \dots, C_{N_p}$ and an n'-ic which satisfies the conditions $C'_1, C'_2, \dots, C'_{N'_p-1}$ make up an $(n+n')$-ic which passes through a point-base $N+N'-1$ contained in $N+N'$ (§ 2). This must pass through the whole of $N+N'$ (§ 3). Hence one of the conditions $C_{N_p+1}, C'_{N'_p}$ must be satisfied (§ 2); and since, by hypothesis, the condition C_{N_p+1} is not necessarily satisfied by the n-ic, the condition $C'_{N'_p}$ is necessarily satisfied by the n'-ic. In the same way it is shown that each of the conditions $C'_{N'_p}, C'_{N'_{p-1}}, \dots, C'_{N'_1}$ is a consequence of those which precede ($N'_p < N'_{p-1} < \dots < N'_1 \leqslant N'$ since $N_p > N_{p-1} > \dots > N_1 \geqslant N$). If at any time the conditions applied to the n'-ic make it vanish identically, then all the remaining conditions up to $C'_{N'}$ are consequences of those already applied. Hence there are at least $p = d_n - D_n$ of the conditions $C'_1, C'_2, \dots, C'_{N'}$ which are consequences of the rest, when applied to an n'-ic. Hence $e'_{n'} \geqslant d_n - D_n$. Similarly $e_n \geqslant d'_{n'} - D_{n'}$.

These two inequalities can be converted into equalities by means of the known formulæ

$$N - e_n + d_n + 1 = \tfrac{1}{2}(n+1)(n+2), \qquad (1)$$

and $D_n + 1 = \left[\tfrac{1}{2}(n-l+1)(n-l+2)\right] + \left[\tfrac{1}{2}(n-m+1)(n-m+2)\right]$, (2)

the square brackets in (2) indicating that the products enclosed are to be retained only when the individual factors are positive. By formula (2), since $n + n' = l + m - 3$, we have

$$D_n + 1 + D_{n'} + 1 = \tfrac{1}{2}(n-l+1)(n-l+2) + \tfrac{1}{2}(n'-l+1)(n'-l+2).$$

Also, by (1), $\quad e'_{n'} - d'_{n'} = N' + 1 - \tfrac{1}{2}(n'+1)(n'+2)$,

and $\quad e_n - d_n = N + 1 - \tfrac{1}{2}(n+1)(n+2)$.

Also $\quad N + N' = lm$.

Adding, we have $(e'_{n'}-d_n+D_n)+(e_n-d'_{n'}+D_{n'}) = 0.$

Therefore $$e'_{n'} = d_n-D_n \quad\text{and}\quad d'_{n'} = e_n+D_{n'}. \tag{3}$$

The two formulæ (3) are equivalent by virtue of (1) and (2).

§ 5. CONSEQUENCES OF THE THEOREM.

The equations $e'_{n'} = d_n-D_n$ and $d'_{n'} = e_n+D_{n'}$,

where $$n' = l+m-n-3,$$

besides giving all the excesses and defects of N' in terms of those of N, yield the following important results, which are different aspects of a single theorem. We assume as heretofore that the whole intersection of C_l, C_m is in the finite region of the plane.

(i.) If $e'_{n'} = 0$, then $d_n = D_n$, *i.e.*, an n-ic through N passes through the whole of $N+N'$; and, if $e'_{n'} > 0$, then $d_n > D_n$. *Hence an n-ic drawn through any point-base N, forming part of the intersection of two curves C_l, C_m, is or is not necessarily of the form $C_l S_{n-l} + C_m S_{n-m}$, according as the $(l+m-n-3)$-ic excess of the point-base N', forming the remaining intersection of C_l, C_m, is or is not zero.*

(ii.) If $e_n = 0$, then $d'_{n'} = D_{n'}$. *Hence, if the n-ic excess of the point-base N is zero, i.e., if the point-base N supplies N independent conditions for an n-ic, or in other words is* regular *for an n-ic, then an $(l+m-n-3)$-ic through N' is of the form $C_l S + C_m S'$. If $l+m-n-3$ is less than m, but not less than l, the $(l+m-n-3)$-ic is of the form $C_l S$; and, if $l+m-n-3$ is less than both l and m, there is no $(l+m-n-3)$-ic through N'.*

(iii.) An important particular case of (ii.) is obtained by putting $n = l$. *If the point-base N, forming part of the intersection of two curves C_l, C_m, is* regular *for an l-ic, then any $(m-3)$-ic through the point-base N', forming the remaining intersection of C_l, C_m, is of the form $C_l S$; and there is no $(m-3)$-ic through N' if $l > m-3$.* (*Cf. loc. cit.*, p. 417, footnote.)

§ 6. POINT-BASE MADE UP OF ONE-SET POINTS.

The reasoning by which the theorem of § 4 is established is applicable, except as regards the use made of Cayley's theorem in formula (2), to any point-base $N+N'$ which consists entirely of one-set points. We thus have the following theorem :—

If a point-base $N+N'$, made up of one-set points, is divided into any two residual point-bases N, N', and if d_n, d'_n, D_n are the n-ic defects, and e_n, e'_n, E_n the n-ic excesses, of N, N', $N+N'$ respectively, then $e'_{n'} \geqslant d_n - D_n$ and $d'_{n'} \leqslant e_n + D_{n'}$, provided any $(n+n')$-ic through any point-base $N+N'-1$ contained in $N+N'$ passes necessarily through the whole of $N+N'$ or vanishes identically.

The greatest value of $n+n'$, consistent with the conditions, gives the best results.

These two inequalities are not equivalent, and cannot be converted into equalities, because Cayley's theorem is no longer applicable. The first gives an inferior limit, and the second a superior limit, for the degree of freedom of a curve of any order through N', if the degrees of freedom of curves of all orders through $N+N'$ and through N are known. By means of formula (1) the inequalities may be written

$$E_n - N' \geqslant e_n - e'_{n'} \geqslant N - E_{n'}.$$

If $N+N'$ is a point-group two other inequalities of some interest can be proved. Those already given may be written

$$D_n \geqslant d_n - e'_{n'} \quad \text{and} \quad D_n \geqslant d'_n - e_{n'},$$

provided an $(n+n')$-ic drawn through all but any one of the $N+N'$ points passes necessarily through the last or vanishes identically. The other inequalities may be written

$$E_n \leqslant e_n + e'_{n'} \quad \text{and} \quad E_n \leqslant e'_n + e_{n'},$$

the first with the proviso that an $(n-n')$-ic can be drawn through the N points without passing through any of the N' points, and the second with the proviso that an $(n-n')$-ic can be drawn through the N' points without passing through any of the N points. It is to be observed that, if n is the same in all four inequalities, n' is different in all except the first two. The greatest value of n', consistent with the conditions, gives in every case the best results.

The N' points supply $d_n - D_n$ independent conditions for an n-ic through the N points. Hence $d_n - D_n$ points can be chosen out of the N' points such that, if an n-ic can be drawn through the N and the $d_n - D_n$ points, it must pass through all the N' points. Now, if an

n'-ic can be drawn through the $d_n - D_n$ points, and an $(n-n')$-ic through the N points, without passing through any of the N' points, then the n'-ic and $(n-n')$-ic make up an n-ic which must pass through all the N' points. The n'-ic must therefore pass through all the N' points, *i.e.*, the N' points do not supply more than $d_n - D_n$ independent conditions for an n'-ic. Also, if no n'-ic can be drawn through the $d_n - D_n$ points, then again the N' points do not supply more than $d_n - D_n$ conditions for an n'-ic. Hence, in all cases, if the proviso respecting the $(n-n')$-ic holds, we have $N' - e'_{n'} \leqslant d_n - D_n$, *i.e.*, $E_n \leqslant e_n + e'_{n'}$, by formula (1). The inequality $E_n \leqslant e'_n + e_{n'}$ is proved in a similar way. The reasoning is not applicable if $N+N'$ is a point-base made up of one-set points unless the residual point-bases N, N' into which it is divided are entirely separate.

§ 7. GENERAL REMARKS.

The following notes will be of some use if they prove suggestive of other possible modes of investigation of the properties of point-bases.

(i.) The most general property of point-bases proved above is that expressed in § 6 by the inequality $e'_{n'} \geqslant d_n - D_n$, which may be interpreted analytically as follows: — Choose the point-base $N+N'$ arbitrarily by giving to each of its one-set points a any situation and any prime equation. The value of α for each one-set point may be found from its prime equation, and the value of $N+N'$ is $\Sigma\alpha$. The conditions supplied by the point-base $N+N'$ for a curve of any order n are known, and the value of E_n is to be found from the fact that $N+N'-E_n$ is the number of independent conditional equations which the point-base $N+N'$ supplies for an n-ic. Let $n+n'$ be the highest value of n for which E_n does not vanish,* *i.e.*, let $n+n'$ be such that $E_{n+n'} > 0$, $E_{n+n'+1} = 0$. Choose arbitrarily a base-point q out of each one-set point a, *i.e.*, choose any of the equations of a as the prime equations of q. Then the point-base N made up of the base-points q is any part of $N+N'$, and the values of N (viz., Σq) and e_n may be found in the same way as those of $N+N'$ and E_n. Let S be the general algebraic curve of an order $>n+n'$ which passes through N, and let $S'_{n'}$ be the general algebraic curve of order n'. Then the number of independent conditional equations (not involving the parameters of S) which must be satisfied by the coefficients of $S'_{n'}$ in order that the composite curve $SS'_{n'}$ may satisfy all the conditions supplied by the point-base $N+N'$ is $N'-e'_{n'}$; and the inequality $e'_{n'} \geqslant d_n - D_n$ states that the number $N'-e'_{n'} \leqslant E_n - e_n$.

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* As n increases E_n diminishes, never remaining stationary. Hence E_n must vanish if n has a sufficiently high finite value. Or, again, e_n must vanish for a given point-base N if e_n vanishes for any point-base which contains N; and a point-base can always be found which contains N and consists entirely of ordinary i-points. For this latter kind of point-base e_n vanishes when n has a sufficiently high value. (C. A. Scott, "A Proof of Noether's Fundamental Theorem," *Mathematische Annalen*, Vol. LII., 1899, pp. 593-597, § 3.)

The validity of the theorem is dependent on the proviso that any $(n+n')$-ic through any point-base $N+N'-1$ contained in $N+N'$ passes of necessity through the whole of $N+N'$, or vanishes identically. This is expressed analytically by saying that, if S is any $(n+n')$-ic such that both $(x-x_1)\,S$ and $(y-y_1)\,S$ satisfy all the conditions supplied by the point-base $N+N'$, then S itself must as a consequence satisfy all the same conditions, or vanish identically, no matter what x_1 and y_1 are. The only pairs of values of x_1, y_1 which need be tried are the coordinates of the points of the point-base.

(ii.) The limitation of a point-base, by definition (§ 1), to the finite region of the plane is important. Base-points which go to infinity disappear, and must be regarded as non-existent, unless some method is devised for taking them into account. If C_l, C_m are any two given curves, having no common factor, the number of conditions satisfied identically by the curve $C_lS_1+C_mS_2$ (S_1, S_2 being arbitrary polynomials) is $lm-d$, where d is the number of (ordinary) points of intersection of the two curves C_l, C_m which have disappeared at infinity. The number $lm-d$ is also the degree of the whole common point-base of the two curves; and the general curve of order n through the whole common point-base is of the form $C_lS_1+C_mS_2$, where, however, S_1, S_2 must be of higher orders than $n-l$, $n-m$ respectively, if n exceeds a certain value.

This suggests the question whether, having given any point-base made up of one-set points, two curves C_l, C_m can always be found which pass through the point-base and have no further intersection (either at the base-points or at any other points) in the finite region of the plane. Any curve through the point-base would then be of the form $C_lS_1+C_mS_2$; but even if, as supposed, two fixed curves C_l, C_m should suffice to determine the point-base uniquely, it does not follow that the fact could be used to much advantage.

(iii.) Since a base-point is the intersection of two or more curves at a point, it is natural to seek a method of obtaining the equations of a base-point at a given point on a given curve.

First, transfer the origin to any ordinary point on the curve. Expand y in powers of x (or x in powers of y if $x=0$ is the tangent to the curve at the origin), and substitute the expansion for y in the power series $\Sigma z_q^p x^{p-q} y^q$, or the expansion for η in $\Sigma z_q^p x^p \eta^q$. We thus obtain a power series in x whose coefficients are linear homogeneous functions of the z's. The coefficient of x^n equated to zero is then the prime equation of a base-point lying on the curve at the origin; and this is a one-set point of degree $n+1$. The independent derived equations are those obtained by equating the coefficients of $x^{n-1}, x^{n-2}, \ldots, x^0$ to zero. There is no definite limit to the degree, since n is unrestricted; but there is only one base-point of assigned degree on the curve at the origin. It should be added that the prime equation $\Sigma \lambda_q^p z_q^p = 0$ is of a simple and special type, all its derivatives $\Sigma \lambda_q^p z_{q-m}^{p-l} = 0$ being deducible from those alone for which $m = 0$, $l = 1, 2, \ldots, n$.

Next, transfer the origin to a multiple point of order i on the curve. By the same process as before, we obtain one prime equation corresponding to each linear branch through the origin. For a superlinear branch of order k (the tangent not being $x=0$) we expand η in integral powers of $x^{1/k}$ and substitute in $\Sigma z_q^p x^p \eta^q$. The coefficients of $x^{n/k}, x^{n-1/k}, \ldots, x^{n-(k-1)/k}$ equated to zero give us k more prime equations, which are not of the same simple type as those corresponding to linear branches. We thus obtain, in all, i independent prime equations, which may be called the fundamental prime equations corresponding to the given curve at the origin. Any equation linearly deducible from these and their derivatives is the prime equation of a one-set point on the curve; and any number of such equations (after omitting those which are deducible from the rest and their derivatives) are the prime equations of a base-point on the curve.

The following presents were made to the Library during the recess:—

"Educational Times," July-Oct., 1900.

"Indian Engineering," Vol. XXVII., Nos. 21-26, May 26-June 30, 1900; Vol. XXVIII., Nos. 1-11, July 7-Sept. 15, 1900.

Rayet, G.—"Observations pluviométriques et thermométriques faites dans le Département de la Gironde de Juin 1898 à Mai 1899," 8vo; Bordeaux, 1899.

Müller, F. — "Vocabulaire Mathématique français-allemand et allemand-français," 1ste Hälfte, 8vo; Leipzig, 1900.

"International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics," 8vo; Toronto, March, 1900.

John Crerar Library.—"Fifth Annual Report for 1899," 8vo; Chicago, 1900.

Fischer, O.—"Der Gang des Menschen," Theil 3, 8vo; Leipzig, 1900.

Cambridge University Library.—"Report of the Library Syndicate for 1899," 4to; Cambridge, 1900.

Extracts from the "Memorie del R. Istituto Lombardo di Scienze e Lettere":—

Veratti, E.—"Ricerche sul Sistema nervoso dei Limax," 4to: Milano, 1900.

Fossati, F.—"Bibliografia Volticina," 4to; Milano, 1900.

Monti, R.—"Le Ghiandole salivari dei Gasteropodi terrestri," 4to; Milano, 1899.

Jatta, M.—"Sulla Genesi della Fibrina," 4to; Milano, 1899.

Adams, J. Couch.—"The Scientific Papers of," Vol. II., 4to; Cambridge, 1900.

Lemoine, E.—"Comparaison géométrografique de douze constructions déduites de onze solutions d'un même problème," pamphlet, 8vo; Paris, 1899.

McGinnis, M. A.—"The Universal Solution for Numerical and Literal Equations," 8vo; London, 1900.

Biddle, D.—"Mathematical Questions and Solutions from the 'Educational Times,'" Vol. LXXIII., 8vo; London, 1900.

Bashforth, F.—"A Second Supplement to a Revised Account of the Experiments made with the Bashforth Chronograph," 8vo; Cambridge, 1900.

Capelli, A.—"Sull' Ordine di Precedenza fra le Operazioni fondamentali dell' Aritmetica," 8vo; Napoli, 1900.

"The Cardioide and some of its related Curves," Inaugural Dissertation zur erlangung der Doctorwürde in University of Strassburg, by R. C. Archibald; 1900.

The following exchanges were received:—

"Annals of Mathematics," Series 2, Vol. I., No. 4; Harvard, 1900.

"Proceedings of the American Philosophical Society," Vol. XXXIX., No. 161; Philadelphia, 1900.

"Proceedings of the Edinburgh Mathematical Society," Vol. XVIII.; 1899-1900.

"Proceedings of the Royal Society of Edinburgh," Vol. XXII.; Edinburgh, 1897-98, 1898-99.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome III., Cah. 2; Tome V., Cah. 1; Procès Verbaux, 1898-1899

"Periodico di Matematica," Anno xvi., Fasc. 1; Livorno, 1900.

"Supplemento al Periodico di Matematica," Anno iii., Fasc. 8; June, 1900.

"Proceedings of the Royal Society," Vol. lxvi., Nos. 430–435, 1900; "Report" and "Further Report to the Malaria Committee," 1900.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxiv., St. 5–8; Leipzig, 1900.

"Bulletin de la Société Mathématique de France," Tome xxviii., Fasc. 2, 3; Paris, 1900.

"Annales de la Faculté des Sciences de Toulouse," Série 2, Tome i., Fasc. 4; 1899.

"Bulletin of the American Mathematical Society," Series 2, Vol. vi., Nos. 9, 10, June, July, 1900; New York.

"Monatshefte für Mathematik und Physik," Jahrgang xi., Pt. 3; Wien, 1900.

"Reale Istituto Lombardo—Rendiconti," Serie 2, Vol. xxxii., 1899.

"Bulletin des Sciences Mathématiques," Tome xxiv., Mars–Juin; Tables des Matières, Tome xxiii.; Paris, 1900.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. vi., Fasc. 5–7, May–July, 1900; Napoli.

"Journal für die reine und angewandte Mathematik," Band cxxii., Heft 3, August, 1900; Berlin.

"Annali di Matematica," Serie 3, Tomo iv., Fasc. 3, 4; Milano, 1900.

"Sitzungsberichte der Physikalisch-medicinischen Societät in Erlangen," Heft 31, 1899; 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. ix., Fasc. 11, 12, and Index; Sem. 2, Vol. ix., Fasc. 1–6; "Anno 297, 1900, Rendiconto dell' adunanza solenne del 10 Giugno 1900 onorata dalla presenza delle LL. Mm. il Re e la Regina," pp. 455–540; Roma, 1900.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," Bd. lii., Nos. 2, 3, 4; 1900.

"Imperial University Calendar," Tokyo, 1899–1900.

"Nyt Tidsskrift for Matematik," Aargang ii., A, Nos. 5, 6; Copenhagen, 1900.

"Revue Semestrielle des Publications Mathématiques," Tome viii., Pt. 2, Oct. 1899–Avr. 1900; Amsterdam, 1899–1900.

"Journal of the Institute of Actuaries," Vol. xxxv., Pt. 4, No. 198; July, 1900.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Vol. xlv., Hefte 1, 2; March, 1900.

"Proceedings of the Physical Society," Vol. xvii., Pt. 2, July, Pt. 3, Sept., 1900.

"Annales de la Faculté des Sciences de Marseille," Tome x., Preface, Fasc. 1–6; 1900.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 1900, Nos. 23–38, May 3–July 26, 1900.

"Proceedings of the Cambridge Philosophical Society," Vol. x., Pt. 6; "Transactions," Vol. xix., Pt. 1; 1900.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. xliv., Pt. 4; 1899–1900.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Kl., 1900, Heft 1, Geschäftliche Mittheilungen, 1900, Heft 1.

"Jahrbuch über die Fortschritte der Mathematik," Bd. xxix., Heft 1, 2, and Jahrgang 1898; 1900.

"Proceedings of the Royal Irish Academy," Vol. v., No. 5; Dublin, 1900.

"Transactions of the Canadian Institute," Vol. vi., Nos. 11, 12, Pts. 1, 2; Toronto, 1899.

"Proceedings of the Canadian Institute," Vol. ii., Pt. 3, No. 9; Feb., 1900.

"Acta Mathematica," Bd. xxiii., Pts. 3, 4; Stockholm, 1900.

APPENDIX (*b*).

(SESSION 1899–1900.)

In connexion with the recent issue of "A List of Members from the Foundation of the Society," it may be of interest to note that the meetings of the Society have been as follow:—From January 16th, 1865, at University College, W.C.; from November 8th, 1866, at the Rooms of the Chemical Society, in Burlington House; and from November 10th, 1870, to the present date, at 22 Albemarle Street, W.

With reference to Mr. Macdonald's communication (pp. 152–157), the following letter has been received from Dr. L. Gegenbauer:—

Soeben erhalte ich das neueste Heft der *Proceedings of the London Mathematical Society*, in welchem Ihre interessante Arbeit, "The Addition Theorem for the Bessel Functions," enthalten ist. Sie beweisen daselbst das von mir in meiner im LXIX. Bande der *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der Kais. Akademie der Wissenschaften in Wien* veröffentlichten Abhandlung, "Über die Bessel'schen Functionen," zuerst aufgestellte Additionstheorem der Bessel'schen Functionen erster Art in der Weise, dass Sie zunächst die Function $\dfrac{J^n(\sqrt{a^2+b^2-2ab\cos\vartheta})}{\{\sqrt{a^2+b^2-2ab\cos\vartheta}\}^n}$ durch ein bestimmtes Integral ausdrücken, sodann einen Factor unter dem Integralzeichen in eine nach den Functionen $C^{\nu}_{\lambda}(\cos\vartheta)$ fortschreitende Reihe entwickeln und endlich die Integration unter Benützung des von

aersetzen werde.

In meiner im LXXXV. Bande der citiert
schienenen Abhandlung, " Das Addition
Functionen, welche bei der Entwicklung von
ungsnennern regulärer Kettenbrüche auftret
neuer Begründung meines Additionstheorem
ohne Beweis mitgetheilt

$$(1)\quad \int_0^\pi e^{ai\cos\phi\cos\psi} J^{\frac{1}{2}(2\nu-1)}(a\sin\phi\sin\psi)\, C_n^\nu(\cos\phi)$$

$$= i^n\sqrt{\frac{2}{a\pi}}\left(\frac{2^{2\nu}\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\nu)}{\Pi(2\nu)}\right)^2 \sin^{\frac{1}{2}(2}$$

aus der speciell folgt:

$$(2)\quad \int_0^\pi e^{ei\cos\phi} J^{\frac{1}{2}(2\nu-1)}(y\sin\phi)\sin^{\frac{1}{2}(2\nu+1)}\phi\, d\phi$$

$$= \sqrt{\frac{2}{\pi y}}\left(\frac{2^{2\nu}\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\nu)}{\Pi(2\nu)}\right)^2 \Big(\sqrt{}$$

Einen Beweis dieser Relation habe ich in meiner
Monatshefte für Mathematik und Physik, public
über die Bessel'schen Functionen erster Art
zugleich hervorgehoben, dass dieselbe zu der En

$$e^{ai\cos\phi\cos\psi} J^{\frac{1}{2}(2\nu-1)}(a\sin\phi\sin\psi)$$

$$= 2^{2\nu-1}\left[\Pi(\nu-1)\right]^2(\sin\phi\sin\psi)^{\frac{1}{2}(2\nu-1)}\sqrt{\frac{2}{a}}$$

Setzt man in der Gleichung (2),

$$x = -a + b\cos\vartheta,$$

$$y = b\sin\vartheta,$$

so verwandelt sich dieselbe in

$$\frac{J^{\nu}\left(\sqrt{a^2+b^2-2ab\cos\vartheta}\right)}{\left\{\sqrt{a^2+b^2-2ab\cos\vartheta}\right\}^{\nu}}$$

$$= \sqrt{\frac{\pi}{2}}\left(\frac{\Pi(2\nu)}{2^{2\nu}\Pi\left(\frac{2\nu-1}{2}\right)\Pi(\nu)}\right)^2 (b\sin\vartheta)^{-\frac{1}{2}(2\nu-1)}\int_0^{\pi} e^{(-a+b\cos\vartheta)i\cos\phi}$$

$$\times J^{\frac{1}{2}(2\nu-1)}(b\sin\vartheta\sin\phi)\sin^{\frac{1}{2}(2\nu+1)}\phi\, d\phi.$$

Entwickelt man nun die Function $e^{bi\cos\vartheta\cos\phi}J^{\frac{1}{2}(2\nu-1)}(b\sin\vartheta\sin\phi)$ mit Hilfe der Gleichung (3) in eine nach den Functionen $C_{\lambda}^{\nu}(\cos\phi)$ fortschreitende Reihe, so erhält man weiter

$$\frac{J^{\nu}\left(\sqrt{a^2+b^2-2ab\cos\vartheta}\right)}{\left\{\sqrt{a^2+b^2-2ab\cos\vartheta}\right\}^{\nu}}$$

$$= \frac{2^{1-2\nu}}{b^{\nu}}\left(\frac{\Pi(2\nu-1)}{\Pi\left(\frac{2\nu-1}{2}\right)}\right)^2 \sum_{\lambda=0}^{\lambda=\infty} i^{\lambda}\frac{(\lambda+\nu)\Pi(\lambda)}{\Pi(\lambda+2\nu-1)}J^{\nu+\lambda}(b)\,C_{\lambda}^{\nu}(\cos\vartheta)$$

$$\times\int_0^{\pi} e^{-ai\cos\phi}C_{\lambda}^{\nu}(\cos\phi)\sin^{2\nu}\phi\, d\phi,$$

woraus sofort das Additionstheorem folgt, weil, wie ich in meiner im LXXV. Bande der citierten *Sitzungsberichte* veröffentlichten Mittheilung "Über die Functionen $C_{\lambda}^{\nu}(x)$" gezeigt habe

$$2^{\nu}i^{\lambda}\Pi(\nu-1)(\lambda+\nu)\,y^{-\nu}J_{(y)}^{\lambda+\nu},$$

der Coëfficient von $C_{\lambda}^{\nu}(x)$ in der Entwicklung von e^{xyi} nach den eben genannten Functionen ist.

Gestatten Sie mir noch, darauf hinzuweisen, dass man unter Benutzung des eben erwähnten Satzes aus Ihrer Gleichung (A) sofort das Integral

$$\int_0^{\pi}\frac{J^{\nu}\left(\sqrt{a^2+b^2-2ab\cos\vartheta}\right)}{\left\{\sqrt{a^2+b^2-2ab\cos\vartheta}\right\}^{\nu}}C_{n}^{\nu}(\cos\phi)\sin^{2\nu}\phi\, d\phi$$

$$= \frac{2^{\nu+1}\nu\Pi(\nu)\Pi(n+2\nu-1)}{\Pi(n)}\left(\frac{2^{\nu}\Pi\left(\frac{2\nu-1}{2}\right)}{\Pi(2\nu)}\right)^2(ab)^{-\nu}J_{(a)}^{n+\nu}J_{(b)}^{n+\nu},$$

ableiten kann, aus welchem sich, wie ich im LXXXV. Band der oft genannten *Sitzungsberichte* bemerkte, unmittelbar das Additionstheorem ergibt.

Sollten Sie glauben, dass die obigen Auseinandesetzungen die Mitglieder der London Mathematical Society interessieren, so ersuche ich Sie, dieselben ihnen im Anschlusse an Ihre Arbeit vorzulegen.

For the accompanying obituary notice of the late Signor Beltrami, the Council are indebted to Prof. G. H. Bryan.

Eugenio Beltrami was born at Cremona on November 16th, 1835. His grandfather, Giovanni Beltrami, was a famous sculptor on precious stones, and his father, Eugenio Beltrami, senior, was also an artist, especially in miniatures, his mother, Elisa, of the Venetian family Barozzi, being still alive. Beltrami inherited the love of fine arts, and was brought up in his youth by his mother and grandfather. He attended school at Cremona, except in 1848–49, when, owing to the war in Lombardy, he went with his mother to Venice. From 1853 to 1856 he studied mathematics under Brioschi at Pavia, where he held a scholarship in the Collegio Ghisleri. His University career was cut short in consequence of an agitation of the students against their Rector, and he was obliged to accept an appointment at Verona as private secretary to Diday, engineer to the Lombardy Railways. For political reasons he lost this post in January, 1859, but was reappointed after the battle of Magenta, when Diday transferred his office to Milan, taking Beltrami with him.

At Milan Beltrami renewed his mathematical studies, devoting much time to reading the works of Bordoni, Brioschi, Monge, Jacobi, Abel, Gauss, Lagrange, and Riemann, and an impetus was given to these studies by his making the acquaintance of Cremona, when that mathematician came to Milan to deliver a course of lectures at the Liceo S. Alessandro.

Beltrami rapidly developed as a mathematician, and in 1862 we find him publishing two papers in the *Annali di Matematica*, edited by Tortolini, of Rome. These proofs of his power resulted in his appointment as Professor Extraordinarius of Algebra and Analytical Geometry in the University of Bologna. His tenure of this chair was of short duration, for in the following year he was appointed Professor Ordinarius of Geodesy at Pisa. Here he remained three years and made the acquaintance of Betti, which developed into a life-long friendship. He also enjoyed the society of Riemann, who was staying in Pisa on account of his health. But the climate did not suit his mother, and, accordingly, Beltrami sought an early opportunity of returning to Bologna, where he lectured on rational mechanics. In February, 1868, he married Amalia Pedrocchi, of Venice, and about the same time published his famous memoir on non-Euclidean geometry, entitled, " Saggio d' interpretazione della geometria non euclidea." In February, 1870, Rome was restored to Italy and became the seat of Government. On the appointment of Antonio Scialoja as Minister of Public Instruction, the University of Rome was reconstituted, and in October, 1873, Beltrami was offered, and accepted, the post of Professor Ordinarius in the subject of Rational Mechanics, with the added duty of conducting a class in Higher Analysis. On entering into office, however, Beltrami found that the scheme of reconstitution was hardly accomplishing its ends in the way that had been predicted; he was alarmed at the difficulties attaching to his new duties, which interfered with his private studies, and he lost no time in seeking a more congenial post. Accordingly, in 1876 he accepted an appointment at Pavia as Professor of Mathematical Physics and Lecturer in Higher Mechanics. His departure from Rome was a source of much regret to his colleagues there, who hoped that he might soon return to them; but this hope was not realized until 1891. At Pavia, if the climate was not healthier, he, at any rate, found greater quiet and new friends, among whom Felice Casorati became his close companion. The premature death of Casorati in 1890 was a source of considerable grief to Beltrami, and was, probably, the factor which determined his return to Rome in the following year. Here he held the office of Councillor of Public Instruction; two years ago he was elected President of the Reale Accademia dei Lincei, and he was elected to the Senatorial rank shortly before his death.

Of Beltrami's works, the catalogue in the special number of the *Atti dei Lincei* for June, 1900, enumerates one hundred and thirty-four. Prof. Cremona, in his obituary address to the Academy, describes Beltrami as a " self-made man " ; but by this he means rather a " self-taught man "—a man of original ideas, in fact. His duties at Pisa directed his attention to Gauss's theory of surfaces and the mathematical theory of maps. He was thus led on to the subject of conformable representation, to transformations of complex variables on surfaces, and, finally, to non-Euclidean geometry; in which latter subject we are largely indebted to Beltrami for the fundamental properties of the pseudosphere, and for many developments of the doctrines of Gauss and Lobatchewsky.

In his obituary address on Brioschi Beltrami distinguished two schools of mathematicians—the classical school, represented by Euler and Jacobi; and another school, founded by Lagrange, and developed by Gauss and Dirichlet, and still further developed by Cauchy and Riemann. Beltrami claimed that Brioschi was one of the first school, and Cremona now places Beltrami himself in the second.

From 1871 to 1874 Beltrami published an extended series of articles on the kinematics of fluids in the *Memorie* of the Bologna Academy, and, subsequently, he devoted considerable attention to the theory of potential, elasticity, electricity and magnetism, conduction of heat, and optics. In the study of elasticity he was led to the conclusion that Lamé's equations were intimately connected with the euclideity of space, and he sought to extend them to spaces of constant curvature. The discoveries of Maxwell, leading to the substitution of stress in a medium for action at a distance, afforded a new field for Beltrami's scientific work, and Hertz's experiments, elaborated in Italy by Righi, lent an additional interest to this study. Instead of starting with Hamilton's generalized equations of dynamics, Beltrami sought to base the equations of the electromagnetic field directly on D'Alembert's principle—a method which, without introducing any substantial modification in Maxwell's theory, led to the required result in a simpler manner. The last period of Beltrami's life was devoted almost entirely to this branch of applied mathematics; yet we find him also writing on thermodynamic potentials, on spherical functions, on Huygens's principle, and as late as 1895 on Lagrange's equations. Beltrami's writings were all characterized by a certain perfection of style and neatness of expression which enabled him to render signal service to mathematicians in clearing up obscure points in the previously existing literature on the branches of mathematics which he made his especial study.

That a certain connexion exists between music and mathematics has been frequently stated. Such an idea was put forward by Sylvester in his paper "On the Real and Imaginary Roots of Equations" (*Phil. Trans.*, 1864). Beltrami was an ardent musician, and in his leisure hours he derived great pleasure from performing on the pianoforte selections from the works of Bach, Mendelssohn, Schumann, and other composers. Prof. Cremona gives extracts from two of Beltrami's letters, one to Dr. Gustav Wolff, Professor at the

Conservatorium of Leipzig, embodying the idea of a relation between music and mathematics, and the other referring to Sylvester's note and to the fact that the composer Meyerbeer had commenced his career with the study of mathematics.

Beltrami sought no distinctions, but these were freely conferred on him. The Academy of Bologna, the Italian Society, the Accademia dei Lincei, the Istituto Lombardo, the Turin Academy, the Royal Society of Naples, the Academies of Berlin, Paris, Munich, Göttingen, and our Society, were glad to include him on their membership lists. The Universities of Kazan and Halle conferred honorary degrees on him; he was also one of the Knights of Savoy (Cavaliere del merito civile di Savoia) from 1879; and it is interesting to learn that Beltrami, in conjunction with Hermite and Bjerknes, were appointed as a commission to report on the claims of Sofia Kowalevsky for the chair which she subsequently held in the University of Stockholm till her death, two years later.

John James Walker, the son of John Walker and Ann (*née* Fricker) his wife, was born at Kennington, Surrey, on October 2nd, 1825.

On the paternal side he was Irish: his great grandfather Matthias, his grandfather John (F.T.C.D.), and his father were all graduates of Trinity College, Dublin, and were remarkable for their combined mathematical and classical tastes. John Walker, junior, appears to have been for a short time (1831) the Headmaster of "University of London School"[*]—a fact which did not come to J. J. Walker's knowledge apparently until the summer of 1899.[†] The connexion with this school soon terminated, and a "London High School" was started at Tavistock House, where, subsequently, Charles Dickens had his residence. Later on Mr. Walker was appointed Headmaster of Plymouth New Grammar School, and here our Mr. Walker entered upon his earliest training, which was carried on entirely under his father's eye.

A short time previous to his father's death, which took place in 1845, Mr. Walker entered Trinity College, Dublin, in 1843. Here he took his degree in 1849, as a Senior Moderator in Mathematics and Physics, having been also an Honourman in Classics frequently

---

[*] *Cf. Alphabetical and Chronological Register for* 1831–1891, p. 9 (University College School, London, by Temple Orme).

[†] As I learn from Mr. W. F. Walker. [R. T.]

[illegible] Mr. Walker removed to London, and
menced his relations with University College
and myself became colleagues in September,
noon work as Lecturer in Applied Mathematics
morning (Pure) Mathematical Master. By this
rarely met at school, but, as we were both ele
Mathematical Society, on October 16th, 186
opportunities of meeting one another. From
Walker was Vice-Principal of University Hall
(in 1874) he married Emma, daughter of th
Turner, of Newcastle, who survives him. His
School terminated in 1888.* Mr. Walker took
work of our Society, being a frequent contributo
and a referee. He was a member of the Coun
1869, to November, 1874, and then again from
November, 1894; he was Vice-President for t
years and was President from November, 1888, tc
Mr. Walker was also a member of the Physi
1883 he was elected a Fellow of the Royal Society

His range of mathematical reading was exten
script memorandum of his states that he contrib
matical papers to the *Philosophical Magazine* ("I
1853), *Cambridge and Dublin Mathematical Journa
of Mathematics*, *Messenger of Mathematics*, *Philos*
(1888), and *Reports of* [illegible]

His earlier work, done when M'Cullagh, Salmon, Graves, and Townsend ruled over Trinity, had a strong bias in the direction of "pure geometry," and his later work tended in the direction of the algebraic methods and subjects. The valedictory address, "Of the Influence of Applied on the Progress of Pure Mathematics" (*Proc.*, XXII., 4–18), is said by Prof. Greenhill, in his valedictory address (*Proc.*, XXIV.), to show "how many of the most abstruse theorems of pure analysis owe their origin to ideas which arose in connexion with concrete and even practical requirements." The subject dealt with in this address was one fertile in interest and importance in its scope, and, doubtless, had Mr. Walker had *carte blanche*, he would, from the stores acquired by a fairly extensive range of reading, have made it even a finer piece of work than it is. The pains evidently taken with it are characteristic of his usual patient care and habit of looking all round his subject.

The following picture of him as a teacher comes from an old pupil:—"My recollections of him and of his 'Natural Philosophy' lectures are among the pleasantest and most vivid remembrances of the old days at University College School. His refinements of explanation, his precision of expression, his grave courtesy of manner—how they all come back! . . . We had excellent opportunities, all unsuspected, of learning science and manners from a master of both. His reproaches were pointed and eloquent, but too nice for the thick-skinned, . . . an excellent teacher for a willing ear."

Mr. W. F. Walker* writes that the general tenour of his brother's life was, throughout, simple and devoid of all exciting events; a life of quiet, industrious work, scholastic tastes, of calm and genial intercourse with family and friends. Underlying a somewhat reserved temperament, not only were his family affections as a son and brother and relative always keen, loyal, and self-sacrificing, but his attachments to, and esteem and confidence from, the early and later friends of his private life always kept up and treasured to the last.

On February 14th, 1900, Mr. Walker appeared to be quite in his usual health; on the 15th he was found insensible in his bath-room,

---

* This gentleman has placed a quantity of interesting matter at my disposal, which I have freely used. He was only ten years of age when the father died, and henceforward our Mr. Walker, then a student, was *in loco parentis* to Mr W. F. Walker.

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,,	XXII.	(from November 1890 to November 1891)		27*s*. 6*d*.
,,	XXIII.	(from November 1891 to November 1892)		20*s*.
,,	XXIV.	(from November 1892 to November 1893)		22*s*.
,,	XXV.	(from November 1893 to November 1894)		22*s*.
,,	XXVI.	(from November 1894 to November 1895)		30*s*.
,,	XXVII.	(from November 1895 to November 1896)		35*s*.
,,	XXVIII.	(from November 1896 to November 1897)		32*s*. 6*d*.
,,	XXIX.	(from November 1897 to November 1898)	Part I.	20*s*.
			Part II.	20*s*.
,,	XXX.	(from November 1898 to March 1899)		20*s*.
,,	XXXI.	(from April 1899 to December 1899)		20*s*.
,,	XXXII.	(from January 1900 to June 1900)		20*s*.